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On the cascade approach to the quantum multiscattering problem

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Abstract. The multiscattering problem is studied in the matrix density formalism. We study how to isolate the quasi-classical degrees of freedom in order to connect them with a cascade approach. The different problems that arise, as well as their possible solutions, are discussed and exemplified with a pion-nucleus model.

PACS. 03.65.Nk Scattering theory – 24.10.Lx Monte Carlo simulations (including hadron and parton cascades and string breaking models) – 24.10.Cn Many-body theory – 25.80.Ls Pion inclusive scattering and absorption

1 Introduction

The multiscattering problem is hard to solve in a fully quantum-mechanical context. Consider, for instance, inclusive pion-nucleus scattering, which will be our model system. Around the $\Delta(3,3)$ resonance several channels are open, besides the elastic one: absorption of the pion, inelastic, single charge exchange and double charge exchange. Some reactions can take place several times for the pion inside the nucleus. All the reactions interfere with each other and their typical reaction probabilities are strongly dependent on the region of the nucleus. Cascade methods [1], which reduce the complicated looking output of the reaction to simple steps, seem appropriate to deal with such a problem. However, cascade methods often involve drastic semi-classical simplifications not completely under control. Typically the reaction probabilities are taken from the free cross-section or at most Pauli blocking is included (and as a consequence are one-body mechanisms) and a classical propagation is used in between two collisions. Even so, the potential ability to embody known physics of the problem and the versatility of the cascade approach makes it interesting to study how it could be improved to systematically include quantum corrections.

Cascade methods have been widely used in the context of heavy-ion collisions, where one wants to describe the dynamical evolution of a large number of particles. In the crudest approach only two-body collisions of free particles are considered, without any mean-field effect [2,3]. Complementary to this is the hydrodynamic approach or the more sophisticated time-dependent Hartree-Fock method, where the mean field is well described but collisions are neglected [4,5]. Both models are merged into microscopic kinetic or transport models. Particularly successful has been the Boltzmann-Uehling-Uhlenbeck (BUU) equation which embodies mean-field effects, two-body collisions and Pauli blocking [6–8]. Transport models go back to Boltzmann and are widely used in physics of plasmas of all kinds [9]. (See [10–18] for reviews on the transport theory approach from different points of view.) In heavy-ion collisions, microscopic transport models are applied not only to nucleons but also to mesons (in particular pions) and resonances [19]. Transport models have also been used directly for studying pion-nucleus reactions [20–26] or other meson-nucleus reactions [27].

In the classical kinetic theory the basic quantity is the distribution function in phase space, $f(\boldsymbol{x},\boldsymbol{p},t)$, which satisfies a transport equation of the type gain-loss as in eq. (1.3) below. It has long been recognized that the Wigner transform [28,29] is the natural way to derive a quantum transport equation [9,11,30,31]. The subject has been brought to a high degree of sophistication, as needed for instance in the description of quark-gluon plasmas, where it has to include thermal effects, be consistent with relativistic invariance as well as covariance under non-Abelian gauge transformations [10,14,32–35]. The field of quantum transport theory is currently quite active and with many open issues, both of conceptual and of practical interest [9].

The mark of cascade models is the classical propagation of the particles between two collisions. This, which at

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first sight may be considered as a drawback, could also be one of their main virtues, for this quasi-classical propagation does indeed exist, furthermore it is dominant between two well-separated successive collisions. Consider, for instance, the following form of the uncertainty principle:

$$|\Delta t| \approx \frac{\hbar}{|E - E_{\text{on-shell}}(\boldsymbol{p})|}\,, \tag{1.1a}$$

$$|\Delta \boldsymbol{x}| \approx \frac{\hbar}{|\boldsymbol{p} - \boldsymbol{p}_{\text{on-shell}}(E)|}\,. \tag{1.1b}$$

These relations show that only on-shell states can propagate a long time or distance, and that these long-lasting states can be taken as classical in the sense that only they survive as $\hbar$ if formally taken to zero. Furthermore, from eqs. (1.1) one finds for large $\Delta t, \Delta \boldsymbol{x}$

$$\left|\frac{\Delta \boldsymbol{x}}{\Delta t}\right| \approx \left|\frac{\mathrm{d}E}{\mathrm{d}\boldsymbol{p}}\right|_{\text{on-shell}}, \tag{1.2}$$

which is the classical Hamilton's equation. This crude argument can be made more precise once a definite prescription is chosen to simultaneously use conjugate variables such as (t, E) or $(\boldsymbol{x}, \boldsymbol{p})$ in the quantum-mechanical context, for instance, Wigner's prescription. Of course, eqs. (1.1) only hold as long as interaction does not take place, but they show the existence of two scales in the multiscattering problem: one due to the mean free path of classical states and the other due to the uncertainty principle, *i.e.*, propagation of virtual states. Multiscale problems usually make trouble to approaches which fail to explicitly include such a feature. It seems more promising to divide the problem into two parts: first, the quasi-local virtual states are integrated out in such a way that the elementary vertices of the microscopic theory are substituted by classical effective N-body quasi-elastic plus absorption probabilities, and second, these probabilities are then used in a cascade approach, where only quasi-classical states show up explicitly.

In order to clarify the meaning of the rather abstract program above, we should place it in the appropriate context. One of our motivations has been to understand and justify the success of the approach in [24,25]. There, all the pion-nucleus inclusive reactions are computed along the following lines: the pion self-energy is computed in nuclear matter including all the Feynman graphs considered relevant for energies around the resonance. The imaginary part of this self-energy is then considered as a pion "width" against decay of the elastic channel into reaction states. With the help of a local density prescription, the total reaction cross-section is then calculated. Furthermore, use is made of Cutkosky rules [36,37] in order to separate the reaction width into the several reaction channels, namely, absorption and quasi-elastic (with or without charge exchange). This information is then used in a Monte Carlo simulation of the path of the pion inside the nucleus: the pion is treated classically in between collisions but the reaction probabilities are computed microscopically from Feynman graphs. Several questions arise from this rather intuitive approach, such as how does it follow from a purely quantum-mechanical calculation?, how can the classical and local density approximations be systematically improved? how to avoid double counting between, say, a genuine three-body absorption mechanism included in the microscopic calculation and a quasi-elastic followed by a two-body absorption coming from the Monte Carlo simulation, having both the same final state? The Monte Carlo simulation for the pion can be cast in the form of a transport equation,

$$\partial_t f(\boldsymbol{x}, \boldsymbol{p}, t) = \int \mathrm{d}^3x'\, \mathrm{d}^3p'\, Q(\boldsymbol{x}, \boldsymbol{p}; \boldsymbol{x}', \boldsymbol{p}') f(\boldsymbol{x}', \boldsymbol{p}', t) - R(\boldsymbol{x}, \boldsymbol{p}) f(\boldsymbol{x}, \boldsymbol{p}, t)\,, \tag{1.3}$$

where $Q(\boldsymbol{x}, \boldsymbol{p}; \boldsymbol{x}', \boldsymbol{p}')$ represents the unit time probability for the transition $(\boldsymbol{x}', \boldsymbol{p}') \to (\boldsymbol{x}, \boldsymbol{p})$ in phase space and $R(\boldsymbol{x}, \boldsymbol{p})$ is the probability of leaving the state $(\boldsymbol{x}, \boldsymbol{p})$. Q contains both elastic and quasi-elastic processes, while R is the total reaction rate. The absorption rate is, thus,

$$A(\boldsymbol{x}, \boldsymbol{p}) = R(\boldsymbol{x}, \boldsymbol{p}) - \int \mathrm{d}^3x'\, \mathrm{d}^3p' Q(\boldsymbol{x}', \boldsymbol{p}'; \boldsymbol{x}, \boldsymbol{p}). \tag{1.4}$$

Then, which are exactly the kernels R, Q, if any, that will produce the same cross-sections as the Schrödinger equation does? Note that we shall actually deal with just one particle cascading through the nucleus so $f(\boldsymbol{x}, \boldsymbol{p}, t)$ is the probability density in phase space rather than the real density as in standard kinetic theory of plasmas. The equation is however formally identical to a transport equation.

In this work we address those issues. Our main concern has been to write exact quantum-mechanical equations in such a way that the connection with cascade methods were immediate. To this end, a density matrix formalism is used plus Wigner's prescription, in order to achieve a well-defined classical limit. The Wigner transformation has been applied before in the literature to study scattering, most notably by Remler in a series of papers [38–41]. Our emphasis is, however, different since we are interested in isolating the N-body quasi-elastic and absorption rates seen by a quasi-classical particle so that the correct quantum results are recovered in a cascade model.

We shall not be concerned here with relativistic invariance (although relativistic kinematics is allowed and actually used in applications involving pions at intermediate energy), thermal effects, or gauge invariance, however, we shall find that some of the findings in these more sophisticated fields are also of interest here. We shall find that a full Wigner transform, in space and also in time, is needed in order to include inelastic channels, a point not usually realized in non-relativistic applications of the Wigner transform to collision theory.

As we have said practical cascade models are often very simplified from the many-body point of view. Typically, one deals with classical particles moving in a mean-field potential, and classical collisions using in-vacuum cross-sections or decaying width in-vacuum lifetimes, restricted by Pauli blocking in the final states. This procedure mimics at a classical level the evolution described by Feynman

graphs. The problem with a direct diagrammatic approach is, of course, that a realistic description would require to carry out the computation to graphs of arbitrarily high orders. The cascade method aims at an efficient procedure to carry out a resummation of those graphs, at the price of a classical description. To improve on this approach it would be interesting to make a formulation using exact relations between resummed sets of graphs, much in the line of the kinetic Kadanoff-Baym or the Schwinger-Dyson equations, but with an explicit $\hbar$-dependence and within a space-time framework, and then find a prescription to integrate out the virtual, non-classical, intermediate states. The improvement of this approach resides in the fact that, once precise definitions for the effective reaction probabilities are given, one is no longer constrained to use in-vacuum estimations for them and, in principle, many-body effects can consistently and systematically be included. This work should be regarded as an attempt in this direction. Within a particular simplified model we try to implement the previous program. In doing so, we introduce ideas, some of them hopefully new, that presumably will be present in future, more systematic, developments.

The paper is organized as follows: in sect. 2 we show that indeed knowledge of the evolution of the density matrix in the Wigner's form, taken as a density of classical projectiles, gives rise to the correct quantum-mechanical cross-section. In sect. 3 the one-particle system is studied with emphasis on its Wigner's form and the classical limit. In sect. 4 the same analysis is carried out for a particle in an optical potential, *i.e.*, for the elastic channel. Section 5 is devoted to the general many-body evolution equation exemplified with a simple pion-nucleus model, with only pions and particle-hole (ph) excitations as physical degrees of freedom. Section 6 shows how the ph degrees of freedom can be removed in order to obtain a purely pionic evolution equation. In sect. 7 the virtual pionic degrees of freedom are identified and integrated out in order to obtain effective N-body quasi-elastic rates. In sect. 8 the actual outcome of our scheme is illustrated in simple cases. Finally in sect. 9 we summarize our conclusions.

2 Cross-section in the simulation approach

In this section we shall assume that the proper simulation procedure has already been carried out (how to do that will be the subject of subsequent sections) and our present purpose is to show that in this case the correct fully quantum-mechanical cross-section is obtained by the usual method. We start by relating the S-matrix in Wigner's representation to the cross-section, and later contact will be taken with the time evolution operator which is closer to an actual simulation procedure.

Let us assume that the Monte Carlo simulation gives us directly the $\hat\rho_{\rm in}$, $\hat\rho_{\rm out}$ relationship

$$\hat\rho_{\rm out} = \hat S\,\hat\rho_{\rm in}\,\hat S^\dagger, \tag{2.1}$$

where $\hat\rho$ is the density matrix and $\hat S$ is the scattering matrix. Then we shall have

$$\rho_{\rm out}(u) = \int \mathrm{d}^6 v\, S(u,v)\rho_{\rm in}(v). \tag{2.2}$$

Here u, v are points in the phase space $(\boldsymbol{x},\boldsymbol{p})$, $\mathrm{d}^6 v = \mathrm{d}^3x_v\,\mathrm{d}^3p_v$, $\rho(u)$ is the density matrix in Wigner's form [28–31]

$$\rho(\boldsymbol{x},\boldsymbol{p}) = \int \mathrm{d}^3 y\, e^{-i\boldsymbol{y}\cdot\boldsymbol{p}/\hbar}\left\langle \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\middle|\hat\rho\middle|\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right\rangle \tag{2.3}$$

and $S(u,v)$ is a real function related to the Wigner's form of $\hat S$ to be interpreted below as the "probability" density of going from v to u in phase space.

The cross-section from some initial state $|i\rangle$ to some final state $|f\rangle$ is (see, for instance, [42])

$$\sigma(i\to f) = \int \mathrm{d}^2 b \left|\langle f|\hat S\, e^{-i\hat{\boldsymbol{P}}\cdot\boldsymbol{b}/\hbar}|i\rangle\right|^2, \tag{2.4}$$

$\boldsymbol{b}$ being the impact parameter vector and $\hat{\boldsymbol{P}}$ the momentum operator. In addition $|i\rangle$, $|f\rangle$ represent *normalized* initial and final states [42]. Using the property of the Wigner's representation

$$\mathrm{tr}\left(\hat X\hat Y\right) = \int \frac{\mathrm{d}^6 u}{(2\pi\hbar)^3} X(u)Y(u) \tag{2.5}$$

the cross-section can be rewritten as

$$\begin{aligned}\sigma(i\to f) = \int \mathrm{d}^2 b\,\mathrm{d}^3 x\,\mathrm{d}^3 p \frac{\mathrm{d}^3x'\,\mathrm{d}^3p'}{(2\pi\hbar)^3}\rho_f(\boldsymbol{x}',\boldsymbol{p}')\\ \times S(\boldsymbol{x}',\boldsymbol{p}';\boldsymbol{x},\boldsymbol{p})\rho_i(\boldsymbol{x}-\boldsymbol{b},\boldsymbol{p})\,.\end{aligned} \tag{2.6}$$

Now for $|f\rangle$ we take a plane wave with momentum $\boldsymbol{p}_f$

$$\rho_f(\boldsymbol{x},\boldsymbol{p}) = (2\pi\hbar)^3\delta(\boldsymbol{p}-\boldsymbol{p}_f). \tag{2.7}$$

This, of course, spoils the dimensional counting, but that will be fixed later:

$$\begin{aligned}\sigma(i\to \boldsymbol{p}_f) = \int \mathrm{d}^2 b\,\mathrm{d}^3 x'\,\mathrm{d}^3 x\,\mathrm{d}^3 p\\ \times S(\boldsymbol{x}',\boldsymbol{p}_f;\boldsymbol{x}+\boldsymbol{b},\boldsymbol{p})\rho_i(\boldsymbol{x},\boldsymbol{p}).\end{aligned} \tag{2.8}$$

The next step is to use that the fine details of the projectile wave function are not relevant, *i.e.* only its momentum distribution matters. This allows us to rewrite (2.8) in the form

$$\begin{aligned}\sigma(i\to \boldsymbol{p}_f) = \int \mathrm{d}^2 b\,\mathrm{d}^3 x'\,\mathrm{d}^3 p S(\boldsymbol{x}',\boldsymbol{p}_f;\boldsymbol{b},\boldsymbol{p})\\ \times\int \mathrm{d}^3 x\rho_i(\boldsymbol{x},\boldsymbol{p}).\end{aligned} \tag{2.9}$$

Equation (2.9) follows from eq. (2.8) provided

$$\boldsymbol{\nabla}_{\boldsymbol{x}}\int \mathrm{d}^2 b\,\mathrm{d}^3 x' S(\boldsymbol{x}',\boldsymbol{p}_f;\boldsymbol{x}+\boldsymbol{b},\boldsymbol{p}) = 0. \tag{2.10}$$

Physically this is clearly true: the integration on $\boldsymbol{b}$ projects out the $\boldsymbol{x}_\perp$-dependence. Furthermore, varying $\boldsymbol{x}_\parallel$ amounts to a change in the initial position of the projectile along the same incoming trajectory, but that only shifts $\boldsymbol{x}'$ which is integrated out. A proof of (2.10) is given in appendix A.

Then for $|i\rangle$ normalized, but with a narrow momentum distribution, we can take

$$\int \mathrm{d}^3x \rho_i(\boldsymbol{x},\boldsymbol{p}) = \rho_i(\boldsymbol{p}) \approx (2\pi\hbar)^3 \delta(\boldsymbol{p}-\boldsymbol{p}_i) \tag{2.11}$$

and so

$$\sigma(\boldsymbol{p}_i \to \boldsymbol{p}_f) = (2\pi\hbar)^3 \int \mathrm{d}^2b\, \mathrm{d}^3x' S(\boldsymbol{x}', \boldsymbol{p}_f; \boldsymbol{b}, \boldsymbol{p}_i). \tag{2.12}$$

The correct dimensions are recovered by integrating $\boldsymbol{p}_f$ within a solid angle $\mathrm{d}\Omega_f$

$$\frac{\mathrm{d}\sigma(\boldsymbol{p}_i \to \hat{\boldsymbol{p}}_f)}{\mathrm{d}\Omega_f} = \int \mathrm{d}p_f p_f^2\, \mathrm{d}^2b\, \mathrm{d}^3x' S(\boldsymbol{x}', \boldsymbol{p}_f; \boldsymbol{b}, \boldsymbol{p}_i). \tag{2.13}$$

This is the desired relationship between cross-section and Monte Carlo output: to obtain $\mathrm{d}\sigma/\mathrm{d}\Omega$ several projectiles with random impact parameter, represented by the integration over $\boldsymbol{b}$, should be thrown against the target, the simulation procedure (contained in the function $S(\boldsymbol{x}', \boldsymbol{p}_f; \boldsymbol{b}, \boldsymbol{p}_i)$) will put them in $(\boldsymbol{x}', \boldsymbol{p}_f)$ after the interaction, but only the scattering angle is relevant (represented by the integration over $\boldsymbol{x}'$ and p_f). $S(\boldsymbol{x}', \boldsymbol{p}_f; \boldsymbol{b}, \boldsymbol{p}_i)$ can then be identified with the density probability of going from $(\boldsymbol{b}, \boldsymbol{p}_i)$ to $(\boldsymbol{x}', \boldsymbol{p}_f)$ due to the interaction. It can be noted, however, that this "probability density" is not necessarily positive. Positivity is only required for the cross-section, *i.e.*, after integration. Likewise in general, conservation of energy $|\boldsymbol{p}_f| = |\boldsymbol{p}_i|$ is only achieved after integration over $\boldsymbol{b}$ and $\boldsymbol{x}'$.

Another remark is in order regarding (2.13). Unitarity of $\hat{S}$ (or also eq. (2.2)) implies

$$\int \mathrm{d}^6u S(u,v) = 1, \tag{2.14}$$

then a direct use of (2.13) yields

$$\int \mathrm{d}\Omega_f \frac{\mathrm{d}\sigma(\boldsymbol{p}_i \to \hat{\boldsymbol{p}}_f)}{\mathrm{d}\Omega_f} = \int \mathrm{d}^2b = \infty. \tag{2.15}$$

As usual, this means that the non-interacting part of $\hat{S}$ must be removed from the cross-section

$$S(u,v) = Z(u)\delta(u-v) + T(u,v), \tag{2.16}$$

where T is less singular than $\delta(u-v)$ and represents the scattering probability, while $Z(u)$ is the non-interaction probability. T instead of S should be used in (2.13) (whether $\hat{S}$ is unitary or not).

The $\hat{S}$-matrix is a convenient theoretical tool but actually the simulation procedure is more directly related to the evolution operator rather than the $\hat{S}$-matrix: a projectile is sent and the outgoing distribution is observed much later. This is described by

$$\hat{\rho}_{t_2} = e^{-i(t_2-t_1)\hat{H}/\hbar} \hat{\rho}_{t_1} e^{i(t_2-t_1)\hat{H}/\hbar}; \quad -t_1, t_2 \to +\infty. \tag{2.17}$$

In Wigner's form we then have

$$\rho(u,t_2) = \int \mathrm{d}^6v\, U(u,v;t_2-t_1)\rho(v,t_1). \tag{2.18}$$

It has to be shown that for large $t = t_2 - t_1$, $U(u,v;t)$ does the same job as $S(u,v)$ did in (2.13). For large enough t

$$e^{-i(t_2-t_1)\hat{H}/\hbar} \approx e^{-it_2\hat{H}_0/\hbar} \hat{S} e^{it_1\hat{H}_0/\hbar} = e^{-it\hat{H}_0/\hbar}\hat{S}, \tag{2.19}$$

where $\hat{H}_0$, $\hat{H}$ represent the free and full Hamiltonians, respectively, and we have used the well-known property $[\hat{H}_0, \hat{S}] = 0$ [42,43]. This allows to relate the two functions $S(u,v)$ and $U(u,v;t)$ as in eq. (A.2), namely,

$$U(\boldsymbol{x}', \boldsymbol{p}'; \boldsymbol{x}, \boldsymbol{p}; t) \approx \int \frac{\mathrm{d}^3y\, \mathrm{d}^3q}{(2\pi\hbar)^3} e^{i\boldsymbol{q}\cdot(\boldsymbol{x}'-\boldsymbol{y})/\hbar - i\,(H_0(\boldsymbol{p}'+\frac{1}{2}\boldsymbol{q}) - H_0(\boldsymbol{p}'-\frac{1}{2}\boldsymbol{q}))t/\hbar} \times S(\boldsymbol{y}, \boldsymbol{p}'; \boldsymbol{x}, \boldsymbol{p}). \tag{2.20}$$

Now, if U is used in (2.13) instead of S, the $\boldsymbol{x}'$ integration gives $\boldsymbol{q} = 0$ and the result is the same, that is,

$$\frac{\mathrm{d}\sigma(\boldsymbol{p}_i \to \hat{\boldsymbol{p}}_f)}{\mathrm{d}\Omega_f} = \int \mathrm{d}p_f p_f^2\, \mathrm{d}^2b\, \mathrm{d}^3x' U(\boldsymbol{x}', \boldsymbol{p}_f; \boldsymbol{b}, \boldsymbol{p}_i; t). \tag{2.21}$$

Once again, the non-interaction probability is to be removed from U. Also note that the result does not depend on t since $\boldsymbol{q}$ vanishes in (2.20) after $\boldsymbol{x}'$ integration. This requires t to be large so that (2.19) holds.

Let us summarize the outcome of this section. A simple-minded method to obtain the cross-section would be: 1) to compute the kernel of the evolution equation, $U(u,v;t)$; 2) to use it in a simulation procedure as a transition probability density from the point v in phase space at time t_1 to the point u at t_2; 3) to count the outgoing particles to extract the cross-section as in a real experiment. What has been shown here is that the naive expectation is indeed correct. Note that we use "simulation procedure" to mean a method solving an equation like (2.18). The fact that $U(u,v;t)$ will not be positive definite in general can be a technical problem but the equation itself is well defined.

3 The single-particle evolution equation

In this section we review the simplest case of a one-particle system and its connection with a classical description.

Let us construct eq. (1.3) for a single-particle system, with Hamiltonian $\hat{H}$. We start with the density matrix evolution equation

$$i\hbar \frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \left[\hat{H}, \hat{\rho}\right] \tag{3.1}$$

and rewrite it using the Wigner transformation, as defined in (2.3), to eventually consider its classical limit. The product of two operators can be dealt with by means of the identity

$$\begin{aligned}\big(\hat{A}\hat{B}\big)(\boldsymbol{x},\boldsymbol{p}) = \int \frac{\mathrm{d}^3y\,\mathrm{d}^3q}{(\pi\hbar)^3}\frac{\mathrm{d}^3z\,\mathrm{d}^3k}{(\pi\hbar)^3}A(\boldsymbol{y},\boldsymbol{q})B(\boldsymbol{z},\boldsymbol{k})\\ \times e^{i2(\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{x}/\hbar}\\ \times e^{i2(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{y}/\hbar}e^{i2(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{z}/\hbar},\end{aligned} \tag{3.2}$$

where the left-hand side stands for the Wigner's form of $\hat{A}\hat{B}$ at $(\boldsymbol{x},\boldsymbol{p})$. This can be written more compactly using the notation (simplectic scalar product)

$$\begin{aligned}u\wedge v = -v\wedge u = \boldsymbol{x}\cdot\boldsymbol{q} - \boldsymbol{y}\cdot\boldsymbol{p},\\ u=(\boldsymbol{x},\boldsymbol{p}),\quad v=(\boldsymbol{y},\boldsymbol{q}),\end{aligned} \tag{3.3}$$

as

$$\big(\hat{A}\hat{B}\big)(u)=\int \frac{\mathrm{d}^6v}{(\pi\hbar)^3}\frac{\mathrm{d}^6w}{(\pi\hbar)^3}A(v)B(w)e^{i2(v-u)\wedge(w-u)/\hbar}. \tag{3.4}$$

In order to study the classical limit, we should transform this expression into one with better properties as $\hbar$ goes to zero. To this end we use the identity (in one dimension)

$$e^{ixp/\hbar} = 2\pi\hbar\, e^{i\hbar\partial_x\partial_p}\delta(x)\delta(p), \tag{3.5}$$

where the exponential in the right-hand side is to be expanded as a series of powers of $\hbar$. This identity can be established by considering both sides as distributions on e^{iax} as test function, for arbitrary a. It can immediately be extended to any number of dimensions, and also

$$e^{iu\wedge v/\hbar} = (2\pi\hbar)^6\, e^{i\hbar\partial_u\wedge\partial_v}\delta(u)\delta(v). \tag{3.6}$$

Then (3.4) can be cast in the form

$$\begin{aligned}\big(\hat{A}\hat{B}\big)(u) &= \int \frac{\mathrm{d}^6v}{(\pi\hbar)^3}\frac{\mathrm{d}^6w}{(\pi\hbar)^3}A(u+v)B(u+w)(\pi\hbar)^6\\ &\quad e^{\frac{1}{2}i\hbar\partial_v\wedge\partial_w}\delta(v)\delta(w)\\ &= e^{\frac{1}{2}i\hbar\partial_v\wedge\partial_w}A(u+v)B(u+w)\Big|_{v=w=0},\end{aligned} \tag{3.7}$$

or simply [44]

$$\big(\hat{A}\hat{B}\big)(u) = e^{\frac{1}{2}i\hbar\partial_u^{(A)}\wedge\partial_u^{(B)}}A(u)B(u). \tag{3.8}$$

Expanding in powers of $\hbar$ yields

$$\big(\hat{A}\hat{B}\big)(u) = A(u)B(u) + \frac{i\hbar}{2}\{A(u),B(u)\}_{\mathrm{P}} + \cdots. \tag{3.9}$$

The zero-th order shows that operators commute in the classical limit and the first correction introduces the usual Poisson bracket, $\{A,B\}_{\mathrm{P}} = \partial A\wedge\partial B$.

The evolution equation for $\rho(u,t)$, eq. (3.1), takes the form

$$\begin{aligned}i\hbar\frac{\partial\rho(u,t)}{\partial t} &= \Big(e^{\frac{1}{2}i\hbar\partial^{(H)}\wedge\partial^{(\rho)}} - e^{\frac{1}{2}i\hbar\partial^{(\rho)}\wedge\partial^{(H)}}\Big)\\ &\quad\times H(u)\rho(u,t) \qquad (3.10a)\\ &= 2i\sin\left(\frac{1}{2}\hbar\partial^{(H)}\wedge\partial^{(\rho)}\right)H(u)\rho(u,t). \qquad (3.10b)\end{aligned}$$

At lowest order in $\hbar$ we find

$$\frac{\partial\rho}{\partial t} = \{H,\rho\}_{\mathrm{P}} + \mathcal{O}(\hbar^2), \tag{3.11}$$

which is the classical equation of evolution in phase space. This equation is also referred to as the Liouville equation in mechanics, the Vlasov equation in plasma physics (particularly when the particles move coupled to an electromagnetic field), or (collisionless) transport equation in the context of kinetic theory. Correspondingly, eq. (3.10b) is the quantum transport equation. The approach of using the Wigner transformation to derive a quantum transport equation has become standard [31] and has been extended in various ways, including finite-temperature [14,32] relativistic treatments [45,46], Abelian and non-Abelian gauge covariant definitions of the Wigner function as well as second quantization definitions of a Wigner operator [10].

For subsequent developments, it will be convenient to consider also the more general case of a non-Hermitian Hamiltonian $\hat{H} = \hat{H}_{\mathrm{R}} + i\hat{H}_{\mathrm{I}}$. The new evolution equation is

$$i\hbar\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}^\dagger, \tag{3.12}$$

and the corresponding equation in Wigner's form would be

$$\begin{aligned}&\partial_t\rho(u,t)\\ &\quad= \frac{2}{\hbar}\cos\left(\frac{1}{2}\hbar\partial^{(H)}\wedge\partial^{(\rho)}\right)H_{\mathrm{I}}(u)\rho(u,t)\\ &\qquad+\frac{2}{\hbar}\sin\left(\frac{1}{2}\hbar\partial^{(H)}\wedge\partial^{(\rho)}\right)H_{\mathrm{R}}(u)\rho(u,t)\\ &\quad= \frac{2}{\hbar}H_{\mathrm{I}}(u)\rho(u,t) + \{H_{\mathrm{R}}(u),\rho(u,t)\}_{\mathrm{P}} + \mathcal{O}(\hbar).\end{aligned} \tag{3.13}$$

Throughout this section we have used the classical limit in a rather formal manner. The problem of fixing the $\hbar$-dependence in an expression is not a trivial or even well-defined one. In practice, our point of view has been to adopt the Wigner's form of an operator as that with a smooth classical limit or plainly as that without any $\hbar$-dependence at all. However, it is clear that this cannot be true for *all* operators[1]. After all $\hbar$ is not small, it is rather unity in natural units. The classical limit should be understood as a physical limit of small fluctuations. Depending of the system this may correspond to large times or distances, large momenta or energies, weak coupling (or sometimes strong coupling), low densities or large number of degrees of freedom, among others.

[1] Assume, for instance, that $\rho(\boldsymbol{x},\boldsymbol{p})$, constructed out of some $\hat{\rho}$, is non-zero only in two localized regions of the phase space such that in one of them it is positive and is negative in the other. The positive region should be larger than $\mathcal{O}((2\pi\hbar)^3)$ due to the uncertainty principle, while the negative one can be at most of size $\mathcal{O}((2\pi\hbar)^3)$ since $\hat{\rho}$ is a positive operator. If we were to reconstruct $\hat{\rho}$ using a different value for $\hbar$ it could not be very different in order to satisfy both constraints.

4 Evolution equation and optical potential

In this section, we extend the study of the evolution equation for the density matrix and its classical limit by including some many-body effects by means of an optical potential. Let us consider the following energy-dependent Hamiltonian:

$$\hat{H}(E) = \hat{H}_0 + \hat{V}_{\rm opt}(E), \tag{4.1a}$$

$$\hat{V}_{\rm opt}(E) = -\int \frac{{\rm d}E'}{\pi} \frac{\hat{V}_{\rm I}(E')}{E - E' + i\eta}\,. \tag{4.1b}$$

H_0 is energy independent and Hermitian, and contains the free and Hartree-Fock pieces of the particle self-energy. On the other hand, $\hat{V}_{\rm opt}(E)$ is energy dependent and hence non-Hermitian and non-instantaneous, and contains the intermediate states not in the elastic channel. $i\hat{V}_{\rm I}$ denotes the anti-Hermitian part of $\hat{V}_{\rm opt}$. This is the absorptive part of the optical potential and is non–positive-definite.

The Schrödinger equation in this case takes the form

$$E|E\rangle = \hat{H}(E)|E\rangle, \tag{4.2}$$

or in time representation

$$i\hbar\partial_t|\psi,t\rangle = \int {\rm d}\tau\, \hat{h}(\tau)|\psi,t-\tau\rangle, \tag{4.3}$$

where

$$|\psi,t\rangle = \int \frac{{\rm d}E}{2\pi\hbar} e^{-iEt/\hbar}\psi(E)|E\rangle, \tag{4.4}$$

$$\begin{aligned}\hat{h}(\tau) &= \int \frac{{\rm d}E}{2\pi\hbar} e^{-iE\tau/\hbar}\hat{H}(E)\\ &= \delta(\tau)\hat{H}_0 + \frac{i}{\pi\hbar}\theta(\tau)\int {\rm d}E\, e^{-iE\tau/\hbar}\hat{V}_{\rm I}(E)\,.\end{aligned} \tag{4.5}$$

The non-locality in time in (4.3) comes from non-elastic intermediate states represented by the optical potential. As in the previous section, we would like to construct an evolution equation for $\hat{\rho}(t) = |\psi,t\rangle\langle\psi,t|$, however, it is easy to see that $\hat{\rho}(t)$ does not satisfy an autonomous equation[2]. The problem is that $\hat{\rho}(t)$ does not actually contain the same information as $|\psi,t\rangle$ does; it loses track of the phases. This would be irrelevant if $|\psi,t\rangle$ were the wave function of the whole system, but it is not: it only describes the elastic part. Instead, we have to consider the more general set of operators

$$\hat{\rho}(t,t') = |\psi,t\rangle\langle\psi,t'|, \tag{4.6}$$

as well as their Wigner's transformed (in time energy)

$$\hat{\rho}(t,E) = \int {\rm d}\tau\, e^{iE\tau/\hbar}\hat{\rho}\left(t+\frac{1}{2}\tau, t-\frac{1}{2}\tau\right), \tag{4.7}$$

which are Hermitian. Note that the equal-time density matrix is recovered through energy integration

$$\hat{\rho}(t) = \int \frac{{\rm d}E}{2\pi\hbar}\hat{\rho}(t,E). \tag{4.8}$$

Similarly, we define the Wigner transform of the Hamiltonian $\hat{H}$ (with $\hat{H}(t,t') := \hat{h}(t-t')$)

$$\hat{H}(t,E) = \hat{H}(E). \tag{4.9}$$

In order to write an evolution equation for $\hat{\rho}(t,E)$, it is more convenient to start with the energy representation

$$E|E\rangle = \hat{H}(E)|E\rangle, \tag{4.10a}$$

$$\hat{\rho}(E_1,E_2) = |E_1\rangle\langle E_2|, \tag{4.10b}$$

so that

$$E_1\hat{\rho}(E_1,E_2) = \hat{H}(E_1)\hat{\rho}(E_1,E_2), \tag{4.11a}$$

$$E_2\hat{\rho}(E_1,E_2) = \hat{\rho}(E_1,E_2)\hat{H}^\dagger(E_2). \tag{4.11b}$$

Subtracting both equations and using the notation $E = \frac{1}{2}(E_1+E_2)$, $\omega = E_1 - E_2$ and $\hat{\rho}(\omega,E) = \hat{\rho}(E_1,E_2)$, we obtain

$$\begin{aligned}\omega\hat{\rho}(\omega,E) = \hat{H}\left(E+\frac{1}{2}\omega\right)\hat{\rho}(\omega,E)\\ -\hat{\rho}(\omega,E)\hat{H}^\dagger\left(E-\frac{1}{2}\omega\right),\end{aligned} \tag{4.12}$$

which can also be written in terms of $\hat{\rho}(t,E)$ by Fourier transforming ω:

$$\begin{aligned}i\hbar\partial_t\hat{\rho}(t,E) = e^{\frac{1}{2}i\hbar\overleftarrow{\partial}_t^{(\rho)}\partial_E^{(H)}}\hat{H}(E)\hat{\rho}(t,E)\\ -e^{-\frac{1}{2}i\hbar\partial_t^{(\rho)}\overrightarrow{\partial}_E^{(H)}}\hat{\rho}(t,E)\hat{H}^\dagger(E).\end{aligned} \tag{4.13}$$

This is the evolution or *transport equation* in the presence of an optical potential. It is an autonomous equation for $\hat{\rho}(t,E)$ as a function of t because, as a consequence of energy conservation, no energy derivatives of $\hat{\rho}$ appear in it. Note that if we take the "classical" limit in the right-hand side by taking the explicit $\hbar \to 0$, eq. (3.12) is recovered.

Equation (4.13) has been written using a partial Wigner form, namely, in time energy. It can be further expanded by using the full space-time Wigner's transformation (in time energy and position momentum). This yields the following more symmetrical form of the transport equation which generalizes eq. (3.10a):

$$\begin{aligned}i\hbar\partial_t\rho(u) = e^{-\frac{1}{2}i\hbar\overleftarrow{\partial}_u^{(H)}\wedge\overrightarrow{\partial}_u^{(\rho)}}H(u)\rho(u)\\ -e^{\frac{1}{2}i\hbar\overleftarrow{\partial}_u^{(H)}\wedge\overrightarrow{\partial}_u^{(\rho)}}\rho(u)H^\dagger(u),\end{aligned} \tag{4.14}$$

where now u denotes $(t,\boldsymbol{x};E,\boldsymbol{p})$, and $u\wedge v = tE' - Et' - \boldsymbol{x}\cdot\boldsymbol{p}' + \boldsymbol{p}\cdot\boldsymbol{x}'$. This formula holds actually for a general non-conservative and non-instantaneous Hamiltonian $\hat{H}(t,E)$. In what follows we will not, in general, expand the formulas by expliciting the position momentum part of the Wigner transform.

[2] To see this, consider arbitrarily changing the phases of $|\psi,t\rangle$ for all times t before some t_0. Clearly this manipulation does not affect $\hat{\rho}(t)$ for $t < t_0$, however, due to the non-locality in time in (4.3), the evolution of $|\psi,t\rangle$ and $\hat{\rho}(t)$ will be modified after t_0. This shows that $\hat{\rho}(t)$ does not satisfy an autonomous evolution equation.

As it stands, (4.13) is of little usefulness since it is non-local in time and so all the time derivatives of $\hat\rho$ will contribute to the first one. Fortunately, the expansion in powers of $\hbar$ can be used to bypass this undesirable feature. First let us expand eq. (4.13) using the separation $\hat H(E) = \hat H_0 + \hat V_{\rm R}(E) + i\hat V_{\rm I}(E)$,

$$\begin{aligned} i\hbar\partial_t\hat\rho = {}& \big[\hat H_0 + \hat V_{\rm R}, \hat\rho\big]_- + \big[i\hat V_{\rm I}, \hat\rho\big]_+ \\ &+\left[\partial_E\hat V_{\rm R}, \frac{i\hbar}{2}\partial_t\hat\rho\right]_+ + \left[i\partial_E\hat V_{\rm I}, \frac{i\hbar}{2}\partial_t\hat\rho\right]_- \\ &+\frac{1}{2}\left[\partial_E^2\hat V_{\rm R}, \left(\frac{i\hbar}{2}\right)^2\partial_t^2\hat\rho\right]_- \\ &+\frac{1}{2}\left[i\partial_E^2\hat V_{\rm I}, \left(\frac{i\hbar}{2}\right)^2\partial_t^2\hat\rho\right]_+ + \cdots, \end{aligned} \tag{4.15}$$

where $[\,,\,]_-, [\,,\,]_+$ stand for commutator and anticommutator, respectively. Then, let us demand that there exist a classical limit at all for $\partial_t\rho(\boldsymbol{x},\boldsymbol{p};t,E)$. The first term in the right-hand side of (4.15) poses no problem because it is of order $\hbar$ due to the commutator (cf. (3.9)). Analogously the third- and higher-order terms have explicit $\hbar$ in them. So, we will require

$$\hat V_{\rm I} = \mathcal{O}(\hbar) \tag{4.16}$$

for $\partial_t\rho$ to exist in the limit $\hbar\to 0$. Note that this implies $\hat V_{\rm R} = \mathcal{O}(\hbar)$ too, due to (4.1b). Under this assumption, it follows that the first and the second terms in eq. (4.15) are $\mathcal{O}(\hbar)$. The third term is $\mathcal{O}(\hbar^2)$. The fourth term is $\mathcal{O}(\hbar^3)$, and the others are $\mathcal{O}(\hbar^4)$ and $\mathcal{O}(\hbar^3)$. In general, higher-order time derivatives of $\hat\rho$ appear only at higher order in $\hbar$. This allows to express $\partial_t\hat\rho$ in terms of $\hat\rho(t,E)$ only (without time derivatives) at any given order in $\hbar$: $\partial_t\hat\rho$ in the third and fourth terms are eliminated using (4.15) recursively, $\partial_t^2\hat\rho$ in the fifth and sixth terms are eliminated by applying $(i\hbar\partial_t)$ once to (4.15), etc. Explicitly, through second order, we obtain

$$\begin{aligned} i\hbar\partial_t\hat\rho = {}& \big[\hat H_0,\hat\rho\big]_- + i\big[\hat V_{\rm I},\hat\rho\big]_+ + \big[\hat V_{\rm R},\hat\rho\big]_- \\ &+\frac{1}{2}\big[\partial_E\hat V_{\rm R}, \big[\hat H_0,\hat\rho\big]_- + i\big[\hat V_{\rm I},\hat\rho\big]_+\big]_+ + \mathcal{O}\big(\hbar^3\big). \end{aligned} \tag{4.17}$$

This expansion is expected to be only asymptotic. In what follows we will not, in general, explicitly expand the equations to put them in a manifestly instantaneous form, as done here, but this procedure can be carried out if needed.

It is worth noticing that the real part of the optical potential $\hat V_{\rm R}$, only appears at higher order than the free Hamiltonian $\hat H_0$ or the absorptive part of $\hat V_{\rm I}$. In many cascade calculations only $\hat V_{\rm I}$ is used, (*i.e.*, the cross-section). Equation (4.17) indicates that this is a kind of classical approximation.

As we have seen, the relation (4.16) is needed in order for eq. (4.17) to make sense. Its origin is clearer if $\hat V_{\rm opt}$ is considered as a self-energy in many-body language [47]. From eq. (4.1) the Hartree pieces are included in $\hat H_0$ and then $\hat V_{\rm opt}$ contains only self-energy pieces with loops, the only diagrams with imaginary part. As shown in [48], each loop gives a further power in $\hbar$ to a diagram, consistently with eq. (4.16).

Let us consider the propagation of a particle in infinite nuclear matter. Due to translational invariance, all relevant operators are functions of the momentum and commute among them (internal degrees of freedom are neglected here). In this case (4.17) becomes

$$\partial_t\rho(t,E) = -\frac{\Gamma(E)}{\hbar}\rho(t,E), \tag{4.18}$$

where $\Gamma(E) = \mathcal{O}(\hbar)$ is the width and is given as a power series in $\hbar$. A closed form is more easily obtained directly from (4.12)

$$-i\Gamma(E) = H\Big(E - \frac{1}{2}i\Gamma\Big) - H^\dagger\Big(E + \frac{1}{2}i\Gamma\Big). \tag{4.19}$$

From here, expanding in $\hbar$, the well-known quasi-particle result [47]

$$\Gamma(E) \approx -2Z(E)V_{\rm I}(E), \tag{4.20a}$$

$$Z(E) = (1 - \partial_E V_{\rm R}(E))^{-1} \tag{4.20b}$$

appears at order $\hbar^2$.

It can be noted that from the two equations (4.11) we have extracted only one equation, namely, (4.12). Taking the semi-sum instead of the difference in (4.11), one obtains

$$\begin{aligned} E\hat\rho(t,E) = {}& \frac{1}{2}e^{\frac{1}{2}i\hbar\partial_t^{(\rho)}\partial_E^{(H)}}\hat H(E)\hat\rho(t,E) \\ &+\frac{1}{2}e^{-\frac{1}{2}i\hbar\partial_t^{(\rho)}\partial_E^{(H)}}\hat\rho(t,E)\hat H^\dagger(E), \end{aligned} \tag{4.21}$$

which is a kind of energy shell *constraint equation* [9,10]. At lowest order in $\hbar$, in the full Wigner form, it just says that $E = H(u)$, *i.e.*, the density matrix is on-shell. By construction this equation is consistent with the transport equation (4.13). In fact, using again an asymptotic expansion in $\hbar$, we can put this equation in instantaneous form. The lowest orders are

$$\begin{aligned} E\hat\rho = {}& \frac{1}{2}\big[\hat H_0,\hat\rho\big]_+ + \frac{1}{2}\big[\hat V_{\rm R},\hat\rho\big]_+ + \frac{i}{2}\big[\hat V_{\rm I},\hat\rho\big]_- \\ &+\frac{i}{4}\big[\partial_E\hat V_{\rm I}, \big[\hat H_0,\hat\rho\big]_- + i\big[\hat V_{\rm I},\hat\rho\big]_+\big]_+ + \mathcal{O}\big(\hbar^3\big). \end{aligned} \tag{4.22}$$

If the transport and constraint equations are written as $\partial_t\hat\rho = L_t\hat\rho$ and $E\hat\rho = L_E\hat\rho$, with the linear actions L_t and L_E defined by (4.17) and (4.22), respectively, the compatibility amounts to the statement $[L_t, L_E] = 0$, which can be verified also by explicit calculation to the order shown.

As often emphasized, the two equations, transport and constraint, are needed for a proper description of the evolution [9,10,14,33,34]. In the next sections we will also make use of two equations, however, our approach will involve a different set of kinematic equations, (5.13) and (5.15), or (6.11) and (6.13). This is further discussed at the end of sect. 7.

An equal-time density matrix is often used in non-relativistic transport equations [38]. In the relativistic case the equal-time formulation can also be used [49] but Lorentz invariance is only manifest by using the space-time Wigner transform [9]. As discussed in [33,34] both formulations are equivalent without introducing further approximations, the space-time formulation being nevertheless richer since it contains the energy distribution [33, 34].

The equivalence of the two approaches (equal time *vs.* time energy Wigner distribution functions) holds whenever the Hamiltonian is instantaneous, that is, whenever $H(u)$ is independent of E. This follows from integrating over the energy in (4.14) (a procedure named energy averaging in [33,34]), recalling the relation (4.8) between the two density matrices, and noting that ∂_E acts only on $\rho(u)$ (it is not necessary to assume that the Hamiltonian is conservative, $H(u)$ may depend on t). The situation for a non-instantaneous dynamics is different and it requires the use of the space-time form, as done here. This is not related to relativity but rather to the fact that the particle can leave the elastic channel and spend some time in other inelastic intermediate states before returning to the elastic Hilbert space. In this case to derive an autonomous evolution equation for the equal-time density matrix is no longer straightforward. We will deal with a closely related problem in sect. 7.

5 Non-elastic states

In the previous section we studied the time evolution of the density matrix describing the elastic part of the wave function. Here, we would like to study how to describe the non-elastic part. To fix ideas, consider the scattering of pions by nuclei in the Δ-isobar region [50]. Typical processes are those depicted in fig. 1. Figure 1a shows a self-energy graph contributing to the pion optical potential: the incoming pion π collides with a nucleon of the nucleus which is excited to a Δ-isobar and leaves a hole h in the Fermi sea of nucleons, the Δ couples further with a $N\pi'$ state. If these intermediate states are only virtual, *i.e.*, for a short time of order $\hbar$, the pion π'' emerges with the same energy as π: it is an elastic scattering and it only contributes to the real part of the optical potential. On the other hand, if the intermediate particles $\pi' N h$ are near their mass shell, the $|\pi' N h\rangle$ state can live a long time and we have instead the process in fig. 1b: it is a real decay of a pionic mode into pion-particle-hole and contributes to the imaginary part of the optical, *i.e.*, gives a width to the "elastic" incoming pion π.

We wish to describe such quasi-elastic reactions (fig. 1b) by means of a density matrix formalism appropriate to connect with cascade calculations. In order not to unnecessarily obscure the discussion we shall use a simplified model with two kinds of particles: "pions" and "ph" (particle-hole), both bosons, without explicit Δ-isobar or isospin degrees of freedom. In addition, the equations here will not include pion absorption. The absorption mechanism is developed in appendix D and added later. The

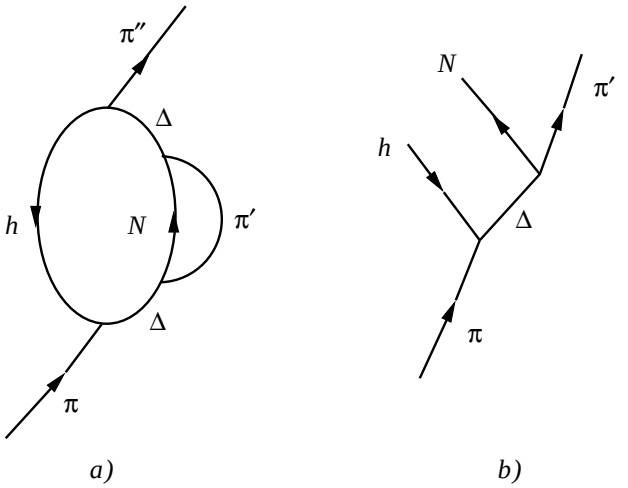

Fig. 1. Typical graphs in Δ-hole model: a) contribution to the pion self-energy; b) a quasi-elastic process.

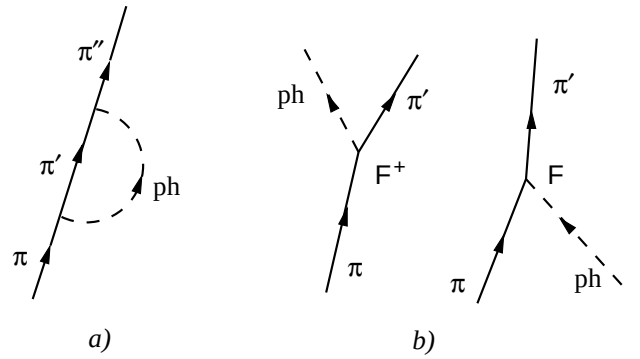

Fig. 2. The same processes as in fig. 1, for the model in eq. (5.1).

model is given by the following Hamiltonian:

$$\hat{H} = \hat{H}_0^{(\pi)} + \hat{H}_0^{(\mathrm{ph})} + \hat{H}_\mathrm{I} := \hat{H}_f + \hat{H}_\mathrm{I}, \tag{5.1}$$

where $\hat{H}_0^{(\pi)}$, $\hat{H}_0^{(\mathrm{ph})}$ are one-body operators for pions and ph, and $\hat{H}_\mathrm{I}$ is the interaction vertex $\pi\pi$ph:

$$\begin{aligned}\hat{H}_\mathrm{I} &= \int \mathrm{d}^3x\, \mathrm{d}^3y\, \mathrm{d}^3z F(\boldsymbol{x};\boldsymbol{y},\boldsymbol{z}) \hat{\phi}_{\mathrm{ph}}(\boldsymbol{x}) \hat{\phi}_\pi^\dagger(\boldsymbol{y}) \hat{\phi}_\pi(\boldsymbol{z}) + \mathrm{h.c.} \\ &:= \hat{F} + \hat{F}^\dagger,\end{aligned} \tag{5.2}$$

and as usual

$$\begin{aligned}\left[\hat{\phi}_\pi(\boldsymbol{x}), \hat{\phi}_\pi^\dagger(\boldsymbol{y})\right] = \left[\hat{\phi}_{\mathrm{ph}}(\boldsymbol{x}), \hat{\phi}_{\mathrm{ph}}^\dagger(\boldsymbol{y})\right] = \delta(\boldsymbol{x}-\boldsymbol{y}), \\ \left[\hat{\phi}_\pi, \hat{\phi}_{\mathrm{ph}}\right] = \left[\hat{\phi}_\pi, \hat{\phi}_{\mathrm{ph}}^\dagger\right] = 0.\end{aligned} \tag{5.3}$$

The interaction vertex is assumed to be elementary (*i.e.*, instantaneous and without energy dependence) but not necessarily local. Within this schematic model, the diagrams of fig. 1 are now described by those in fig. 2.

At $t = -\infty$ the state consists of a single incoming pion. In this model the number of pions is conserved by all pieces of the Hamiltonian. On the other hand, $\hat{H}_f$ preserves the

number of ph's but $\hat{F}^\dagger$ and $\hat{F}$ act creating and deleting one ph, respectively. A generic state of the system will thus contain exactly one pion plus a number k of ph particles, with $0 \le k \le A$, where A is the mass number of the nucleus

$$|\psi\rangle = \sum_{k=0}^{A} \big|\pi,(\mathrm{ph})^k\big\rangle := \sum_{k=0}^{A} |k\rangle. \tag{5.4}$$

$|\pi\rangle = |k=0\rangle$ corresponds to the elastic channel, and $|k\rangle$ is obtained after k inelastic steps. Note that in this model the ph does not have self-energy graphs (unless absorption is included by a vertex πph as in appendix D). The conservation of the number of pions and the free propagation of the ph's are both consequences of the fact that the model does not implement crossing symmetry for the pions (there are no anti-pions in this model). In this sense it has some resemblance with the Lee model [51,52] where the number of possible graphs is severely limited due to the existence of very restrictive conservation laws on the number of particles. In our case the number of graphs is, however, considerably larger since the number of ph's is not restricted: in the one-pion subspace the more general graph consists of the continuous pion line with zero or more outgoing ph lines stemming from it plus zero or more ph internal lines with the ph emitted and reabsorbed by the pion in any order. Since, within the model, a ph cannot couple to a pion-anti-pion pair there are no ph self-energy graphs. This model is devised to describe, in a simplified manner, the problem of pion-nucleus reactions at energies around resonance or below, where pion production is below threshold or barely so. When we introduce absorption in appendix D the pion will couple to a single ph (modeling the absorption of a virtual pion by a nucleon). In this case pions and ph can be transmuted into each other and the number of pions needs not be conserved. (Nevertheless in this region of energies, states with two or more pions can only be virtual and we shall simplify further the exposition by not including them in the formulas.)

Our starting point is the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi,t\rangle = \hat{H}|\psi,t\rangle. \tag{5.5}$$

Using that the states $|k,t\rangle$ are linearly independent, we can write

$$\begin{aligned} i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|k,t\rangle &= \hat{H}_f|k,t\rangle + \hat{F}^\dagger|k-1,t\rangle + \hat{F}|k+1,t\rangle, \\ & k=0,\ldots,A. \end{aligned} \tag{5.6}$$

(Here and in what follows we use the convention that quantities with indices k out of the physical range $0 \le k \le A$ vanish identically. In the present case the terms with $|k=-1\rangle$ and $|k=A+1\rangle$ in the equations for $k=0$ and $k=A$, respectively, are absent.) Although correct, these equations show an unwanted symmetry under time reversal: as the pion goes scattering through the nucleus, the number of ph produced will increase, then we would prefer that states with higher number of ph's had lower ones as a source and not conversely. Let us see how to achieve this in the simple case of $A=2$. Using an energy representation for (5.6):

$$\begin{aligned} E|0,E\rangle &= \hat{H}_f|0,E\rangle + \hat{F}|1,E\rangle, \\ E|1,E\rangle &= \hat{H}_f|1,E\rangle + \hat{F}^\dagger|0,E\rangle + \hat{F}|2,E\rangle, \\ E|2,E\rangle &= \hat{H}_f|2,E\rangle + \hat{F}^\dagger|1,E\rangle. \end{aligned} \tag{5.7}$$

From the last equation $|2,E\rangle = (E-\hat{H}_f+i\eta)^{-1}\hat{F}^\dagger|1,E\rangle$. The time reversal symmetry is broken by choosing $+i\eta$. Substituting in the second equation and using the same method in the resulting equation for $|1,E\rangle$, we end up with

$$\begin{aligned} E|0,E\rangle &= \hat{H}_f|0,E\rangle + \hat{F}\Big(E-\hat{H}_f \\ &\qquad -\hat{F}(E-\hat{H}_f+i\eta)^{-1}\hat{F}^\dagger + i\eta\Big)^{-1}\hat{F}^\dagger|0,E\rangle, \\ E|1,E\rangle &= \hat{H}_f|1,E\rangle + \hat{F}(E-\hat{H}_f+i\eta)^{-1} \\ &\qquad \times\hat{F}^\dagger|1,E\rangle + \hat{F}^\dagger|0,E\rangle, \\ E|2,E\rangle &= \hat{H}_f|2,E\rangle + \hat{F}^\dagger|1,E\rangle. \end{aligned} \tag{5.8}$$

The new equations have the desired form; the equation of motion of $|k\rangle$ involves only the states $|k\rangle$ and $|k-1\rangle$. In general the equations are

$$E|k,E\rangle = \hat{H}_k(E)|k,E\rangle + \hat{F}^\dagger|k-1,E\rangle \tag{5.9}$$

with

$$\hat{H}_k(E) = \hat{H}_f + \hat{F}\hat{G}_{k+1}(E)\hat{F}^\dagger, \tag{5.10a}$$

$$\hat{G}_k(E) = \big(E-\hat{H}_k(E)+i\eta\big)^{-1}. \tag{5.10b}$$

Comparing (5.9) with (4.2), we can see that the operator $\hat{H}_k(E)$ plays the role of an optical Hamiltonian for the state $|\pi(\mathrm{ph})^k\rangle$. The operators $\hat{G}_k(E)$ are the corresponding propagators and in this model they do not contain intermediates states with less than k ph particles. From our previous convention $\hat{H}_{k=A}(E) = \hat{H}_f$ and $\hat{G}_{k=A}(E)$ is just the free propagator. On the other hand, $\hat{G}_{k=0}(E)$ is the full pion propagator and $\hat{H}_{k=0}(E)$ is the full pion optical potential. For this elastic channel the last term in (5.9) is absent and this equation coincides with (4.2).

Upon Fourier transform in time energy eq. (5.9) provides the differential-like evolution equation. It can also be written in the integral form as

$$|k,E\rangle = \hat{G}_k(E)\hat{F}^\dagger|k-1,E\rangle, \quad (k>0). \tag{5.11}$$

Both forms of the evolution equation will be used subsequently.

To proceed, we define the density matrices

$$\hat{\rho}_k(E_1,E_2) := |k,E_1\rangle\langle k,E_2|, \tag{5.12}$$

which satisfy (using (5.11))

$$\begin{aligned} \hat{\rho}_k(E_1,E_2) &= \hat{G}_k(E_1)\hat{F}^\dagger\hat{\rho}_{k-1}(E_1,E_2)\hat{F}\hat{G}_k^\dagger(E_2) \\ &\qquad (k>0). \end{aligned} \tag{5.13}$$

In Wigner's representation, this equation gives $\hat{\rho}_k(t,E)$ in terms of $\hat{\rho}_{k-1}(t,E)$. In order to obtain the differential-like equation for $\hat{\rho}_k(t,E)$ we first rewrite eq. (5.13) as

$$E_1\hat{\rho}_k(E_1,E_2) = \hat{H}_k(E_1)\hat{\rho}_k(E_1,E_2) + \hat{F}^\dagger\hat{\rho}_{k-1}(E_1,E_2)\hat{F}\hat{G}_k^\dagger(E_2) \qquad (5.14)$$

and with the same notation and method used to obtain (4.12) and (4.13), we have

$$\begin{aligned}\omega\hat{\rho}_k(\omega,E) = {} & \hat{H}_k\left(E+\frac{1}{2}\omega\right)\hat{\rho}_k(\omega,E) \\ & -\hat{\rho}_k(\omega,E)\hat{H}_k^\dagger\left(E-\frac{1}{2}\omega\right) \\ & -\hat{G}_k\left(E+\frac{1}{2}\omega\right)\hat{F}^\dagger\hat{\rho}_{k-1}\hat{F} \\ & +\hat{F}^\dagger\hat{\rho}_{k-1}\hat{F}\hat{G}_k^\dagger\left(E-\frac{1}{2}\omega\right), \end{aligned} \qquad (5.15)$$

or in time representation

$$\begin{aligned} i\hbar\partial_t\hat{\rho}_k(t,E) = {} & e^{\frac{1}{2}i\hbar\partial_t^{(\rho)}\partial_E^{(H)}}\hat{H}_k(E)\hat{\rho}_k(t,E) \\ & - e^{-\frac{1}{2}i\hbar\partial_t^{(\rho)}\partial_E^{(H)}}\hat{\rho}_k(t,E)\hat{H}_k^\dagger(E) \\ & - e^{\frac{1}{2}i\hbar\partial_t^{(\rho)}\partial_E^{(G)}}\hat{G}_k(E)\hat{F}^\dagger\hat{\rho}_{k-1}(t,E)\hat{F} \\ & + e^{-\frac{1}{2}i\hbar\partial_t^{(\rho)}\partial_E^{(G)}}\hat{F}^\dagger\hat{\rho}_{k-1}(t,E)\hat{F}\hat{G}_k^\dagger(E). \end{aligned} \qquad (5.16)$$

This can be regarded as an extension of eq. (4.13) to account for the creation of the new $|k\rangle$ states out of $|k-1\rangle$ when $k>0$. Using (5.10a) it is readily verified that this equation preserves unitarity, namely,

$$\partial_t\sum_{k=0}^{A}\mathrm{tr}(\hat{\rho}_k(t,E)) = 0. \qquad (5.17)$$

To summarize this section: we have been able to write an evolution equation for the quantities $\hat{\rho}_k(t,E)$, namely eq. (5.16), which in full Wigner's form provides us with the relationship between simulation-like quantities, $\hat{\rho}_k(\boldsymbol{x},\boldsymbol{p};t,E)$, on the one hand, and microscopic-like quantities, the Green's functions $\hat{G}_k(E)$ and optical Hamiltonians $\hat{H}_k(E)$, on the other, without semiclassical approximations involved. However, this equation is not fully satisfactory because too much information is contained in $\hat{\rho}_k$, namely, all nuclear degrees of freedom as well as far from classical (highly virtual) pionic degrees of freedom. We deal with such problems in the next sections.

6 Removal of ph degrees of freedom

The quantities $\hat{\rho}_k(t,E)$ and their evolution equations contain information both on "pions" and on "ph" states. If we are only interested in the pionic reactions it is convenient to simplify the problem by just working with the pionic degrees of freedom. This is the approach in [24], where the simulation only traces the path of the pion inside of the nucleus. This requires to eliminate the ph degrees of freedom.

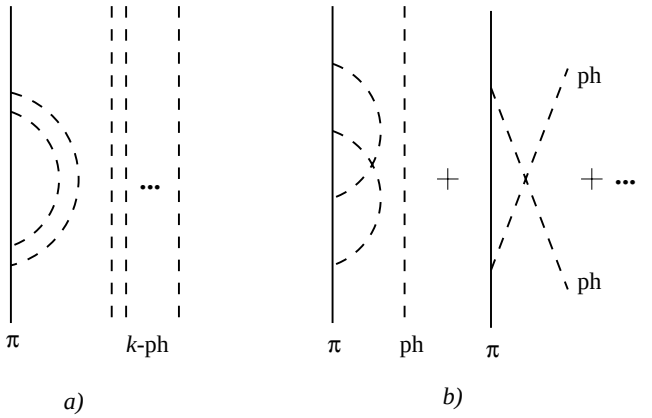

Fig. 3. a) Typical graph contributing to the directed part of $\hat{G}_k$; b) crossed graphs in $\hat{G}_k$ (for $k=1$).

The first idea is to define a new density matrix out of $\hat{\rho}_k$ for the pion only by taking trace over the ph part. If we attempt to do so in eq. (5.13), or in the other equations, we find that this new operator $\mathrm{tr}_{\mathrm{ph}}(\hat{\rho}_k)$ does not obey an autonomous set of equations; the knowledge of $\mathrm{tr}_{\mathrm{ph}}(\hat{\rho}_{k-1})$ does not provide us with $\mathrm{tr}_{\mathrm{ph}}(\hat{\rho}_k)$ because this information is only partial. On the other hand, $\mathrm{tr}_{\mathrm{ph}}$ does not remove the ph energies which are included in E, and in addition, the time t is a common time for the pion and the ph's and that may not be the most appropriate choice. It seems thus necessary to disentangle the different energy and time dependences in $\hat{\rho}_k$ in order to find a density matrix truly depending only on pionic energy and time, as well as $\boldsymbol{x}_\pi$ and $\boldsymbol{p}_\pi$. Likely, the problem is not trivial, and in fact I have only partially succeeded in solving it: there is a solution if the propagators $\hat{G}_k(E)$ only contain direct graphs, as those in fig. 3a, and no crossed terms, fig. 3b. In this case the ph particles are distinguishable and $1,2,\ldots,k$ labels the order in which the ph's have been produced.

When the particles are distinguishable, the total wave function at time t is just the product of individual wave functions taken at the common time t. In energy this corresponds to a convolution. Mathematically, we have

$$\begin{aligned} i\hbar\hat{G}_k(E) = {} & \Big[i\hbar\hat{G}_\pi^{(k)}\circ i\hbar\hat{G}_{\mathrm{ph}}^{(1)}\circ i\hbar\hat{G}_{\mathrm{ph}}^{(2)} \\ & \circ\cdots\circ i\hbar\hat{G}_{\mathrm{ph}}^{(k)}\Big](E) + (\text{crossed terms}), \end{aligned} \qquad (6.1)$$

where the explicit term is that associated to direct graphs, and we have introduced the following notation: the symbol "$\circ$" stands for convolution over the energy dependence

$$\big[A\circ B\big](E) = \int\frac{\mathrm{d}E_A}{2\pi\hbar}\frac{\mathrm{d}E_B}{2\pi\hbar}2\pi\hbar\delta \times(E-E_A-E_B)A(E_A)B(E_B), \qquad (6.2)$$

the operator $\hat{G}_{\mathrm{ph}}^{(\ell)}(E)$ is the free propagator of the ℓ-th ph (as noted previously, the ph particles do not have self-

energy graphs in this model)

$$\hat{G}^{(\ell)}_{\rm ph}(E) = \big(E - \hat{H}^{(\ell)}_{\rm ph} + i\eta\big)^{-1}, \tag{6.3}$$

($\hat{H}^{(\ell)}_{\rm ph}$ is the free Hamiltonian of the ℓ-th ph, included in $\hat{H}^{({\rm ph})}_0$). Finally, the operator $\hat{G}^{(k)}_\pi(E)$ is the pion propagator in the presence of k ph's, but including direct self-energy graphs only. Thus, it is given recursively by (cf. fig. 3a)

$$\hat{G}^{(k)}_\pi(E) = \big(E - \hat{H}^{(k)}_\pi(E) + i\eta\big)^{-1}, \tag{6.4a}$$

$$\hat{H}^{(k)}_\pi(E) = \hat{H}^{(\pi)}_0 - i\hbar\hat{F}\Big[\hat{G}^{(k+1)}_\pi \circ \hat{G}^{(k+1)}_{\rm ph}\Big](E)\hat{F}^\dagger, \tag{6.4b}$$

where $\hat{H}^{(k)}_\pi(E)$ is the optical Hamiltonian of the pion in the presence of k ph's and the operators $\hat{F}, \hat{F}^\dagger$ act only on the $(k+1)$-th ph. Note that $\hat{G}^{(\ell)}_{\rm ph}$ and $\hat{G}_\pi$ act in the ℓ-th ph and pionic Hilbert spaces, respectively and they commute.

The purpose of selecting the direct graphs was to be able to disentangle the energy time and position momentum degrees of freedom carried by each individual particle in the state $\hat{\rho}_k(t,E)$. This is achieved as follows. As shown in detail in appendix B, if crossed terms are dropped in $\hat{G}_k$, a new (more detailed) state $|E_\pi, E_1, \ldots, E_k\rangle_k$ can be defined, depending on π and ph degrees of freedom ($\boldsymbol{x}$ and $\boldsymbol{p}$) and also on their energies, which is related to the state $|k,E\rangle$ in eq. (5.9) by means of

$$\begin{aligned}|k,E\rangle = \int \frac{{\rm d}E_\pi}{2\pi\hbar}\frac{{\rm d}E_1}{2\pi\hbar}\cdots\frac{{\rm d}E_k}{2\pi\hbar}2\pi\hbar\delta \\ \times(E - E_\pi - E_1 - \cdots - E_k) \\ \times|E_\pi, E_1, \ldots, E_k\rangle_k.\end{aligned} \tag{6.5}$$

The new state satisfies the integral-like equation, which is analogous to (5.11),

$$\begin{aligned}|E_\pi, E_1, \ldots, E_k\rangle_k = i\hbar\hat{G}^{(k)}_\pi(E_\pi)i\hbar\hat{G}^{(k)}_{\rm ph}(E_k)\frac{1}{i\hbar} \\ \times\ \hat{F}^\dagger|E_\pi + E_k, E_1, E_2, \ldots, E_{k-1}\rangle_{k-1}, \\ (k>0).\end{aligned} \tag{6.6}$$

Here $\hat{F}^\dagger$ acts on the pion in $|\,\rangle_{k-1}$ and creates the k-th ph, thus building the subset of direct graphs only (and so $|E_\pi, E_1, \ldots, E_k\rangle_k$ is not symmetric under exchange of ph's). Actually this equation is used recursively in appendix B to define the states $|E_\pi, E_1, \ldots, E_k\rangle_k$ starting from the $k=0$ state.

Next, we define the associated density matrix

$$\begin{aligned}\hat{\rho}_k(\omega_\pi, E_\pi, \omega_1, E_1, \ldots, \omega_k, E_k) := \\ \left|E_\pi + \frac{1}{2}\omega_\pi, E_1 + \frac{1}{2}\omega_1, \ldots, E_k + \frac{1}{2}\omega_k\right\rangle_k \\ \left\langle E_\pi - \frac{1}{2}\omega_\pi, E_1 - \frac{1}{2}\omega_1, \ldots, E_k - \frac{1}{2}\omega_k\right|_k,\end{aligned} \tag{6.7}$$

from which $\hat{\rho}_k(\omega, E)$ can easily be recovered by making use of eq. (6.5). This new density matrix satisfies the following recurrence equation (which is a translation of (6.6)):

$$\begin{aligned}&\hat{\rho}_k(\omega_\pi, E_\pi, \omega_1, E_1, \ldots, \omega_k, E_k) \\ &= \hbar^2\hat{G}^{(k)}_\pi\left(E_\pi + \frac{1}{2}\omega_\pi\right)\hat{G}^{(k)}_{\rm ph}\left(E_k + \frac{1}{2}\omega_k\right)\hat{F}^\dagger \\ &\quad\times\hat{\rho}_{k-1}(\omega_\pi + \omega_k, E_\pi + E_k, \omega_1, E_1, \ldots, \omega_{k-1}, E_{k-1}) \\ &\quad\times\hat{F}\hat{G}^{(k)\dagger}_{\rm ph}\left(E_k - \frac{1}{2}\omega_k\right)\hat{G}^{(k)\dagger}_\pi\left(E_\pi - \frac{1}{2}\omega_\pi\right).\end{aligned} \tag{6.8}$$

This equation represents a definite improvement over eq. (5.13) because now the trace can be taken over ph degrees of freedom and a closed set of equations is obtained: on the right-hand side the trace factorizes in the form $\{1,\ldots,k-1\}\{k\}$ and as a consequence ${\rm tr}_{\rm ph}\,\hat{\rho}_k$ is given in terms of ${\rm tr}_{\rm ph}\,\hat{\rho}_{k-1}$.

In order to obtain a pionic density matrix, let us call it $\hat{\rho}^{(k)}(\omega_\pi, E_\pi; E)$, the ph energies E_ℓ can easily be integrated out. Also, the "times" ω_ℓ should be fixed. After Fourier transforming, each ω_ℓ becomes the time t_ℓ at which the ℓ-th ph is detected. In order to preserve unitarity at any "pionic" time t, we should have t smaller that any ph time, so we are lead to choose all the ph times t_ℓ as $+\infty$ in the definition of the purely pionic density matrix, that is,

$$\begin{aligned}\hat{\rho}^{(k)}(t, E_\pi; E) := \lim_{\{t_\ell\}\to+\infty}\int \frac{{\rm d}E_1}{2\pi\hbar}\frac{{\rm d}E_2}{2\pi\hbar}\cdots\frac{{\rm d}E_k}{2\pi\hbar} \\ \times 2\pi\hbar\delta(E - E_\pi - E_1 - \cdots - E_k) \\ \times\,{\rm tr}_{\rm ph}\,\hat{\rho}_k(t, E_\pi; t_1, E_1, \ldots, t_k, E_k).\end{aligned} \tag{6.9}$$

(Where t refers to the pionic time associated to the pionic frequency ω_π. Also we remark that we are taking the limit of large ph times, and not integrating over those times.) This is the density matrix in pionic space describing the pions which have scattered k times (producing k ph's). In particular,

$$\hat{\rho}^{(k=0)}(t, E_\pi; E) = 2\pi\hbar\delta(E_\pi - E)\hat{\rho}_{k=0}(t, E) \tag{6.10}$$

is the density matrix for the elastic channel. Clearly, in $\hat{\rho}^{(k)}(t, E_\pi; E)$, E corresponds to the total energy, which is conserved. On the other hand, E_π is the energy carried by the pion after k collisions and both quantities coincide in the elastic channel. We further discuss on the coexistence of these two energies in $\hat{\rho}^{(k)}(t, E_\pi; E)$ over the end of sect. 7.

As shown in detail in appendix C, if the definition in (6.9) is used in eq. (6.8), the following recurrence is found for $\hat{\rho}^{(k)}$ (analogous to (5.13)):

$$\begin{aligned}\hat{\rho}^{(k)}(\omega_\pi, E_\pi; E) = \hat{G}^{(k)}_\pi\left(E_\pi + \frac{1}{2}\omega_\pi\right) \\ \times\int\frac{{\rm d}E'_\pi}{2\pi\hbar}\,{\rm tr}_{\rm ph}\{2\pi\hbar\delta(E'_\pi - E_\pi - \hat{H}_{\rm ph}) \\ \times\hat{F}^\dagger\hat{\rho}^{(k-1)}(\omega_\pi, E'_\pi; E)\hat{F}\} \\ \times\hat{G}^{(k)\dagger}_\pi\left(E_\pi - \frac{1}{2}\omega_\pi\right),\ (k>0).\end{aligned} \tag{6.11}$$

Moreover, there it is also shown that unitarity is preserved, as a direct mathematical consequence of taking all $t_{\rm ph}$ as $+\infty$, that is,

$$\partial_t \left[\sum_{k=0}^{A} \mathrm{tr} \int \frac{\mathrm{d}E_\pi}{2\pi\hbar} \hat{\rho}^{(k)}(t, E_\pi; E) \right] = 0. \tag{6.12}$$

To obtain the Schrödinger-like equation associated to the integral-like equation in (6.11), we follow the same procedure as that used to obtain eq. (5.15). This gives

$$\begin{aligned}
&\omega_\pi \hat{\rho}^{(k)}(\omega_\pi, E_\pi; E) \\
&\quad = \hat{H}_\pi^{(k)} \left(E_\pi + \frac{1}{2}\omega_\pi \right) \hat{\rho}^{(k)}(\omega_\pi, E_\pi; E) \\
&\qquad - \hat{\rho}^{(k)}(\omega_\pi, E_\pi; E) \hat{H}_\pi^{(k)\dagger} \left(E_\pi - \frac{1}{2}\omega_\pi \right) \\
&\qquad - \mathrm{tr}_{\rm ph} \int \frac{\mathrm{d}E'_\pi}{2\pi\hbar} 2\pi\hbar\delta(E'_\pi - E_\pi - \hat{H}_{\rm ph}) \\
&\qquad \times \left[\hat{G}_\pi^{(k)} \left(E_\pi + \frac{1}{2}\omega_\pi \right) \hat{F}^\dagger \hat{\rho}^{(k-1)}(\omega_\pi, E'_\pi; E) \hat{F} \right. \\
&\qquad \left. - \hat{F}^\dagger \hat{\rho}^{(k-1)}(\omega_\pi, E'_\pi; E) \hat{F} \hat{G}_\pi^{(k)\dagger} \left(E_\pi - \frac{1}{2}\omega_\pi \right) \right]
\end{aligned} \tag{6.13}$$

and then, after Fourier transforming in ω_π,

$$\begin{aligned}
\partial_t \hat{\rho}^{(k)}(t, E_\pi; E) = &\left(\partial_t \hat{\rho}^{(k)} \right)^{(+)}(t, E_\pi; E) \\
&+ \left(\partial_t \hat{\rho}^{(k)} \right)^{(-)}(t, E_\pi; E)
\end{aligned} \tag{6.14}$$

with

$$\begin{aligned}
&\left(i\hbar\partial_t \hat{\rho}^{(k)} \right)^{(+)}(t, E_\pi; E) \\
&\quad = e^{\frac{1}{2} i\hbar \partial_t^{(\rho)} \partial_{E_\pi}^{(H)}} \hat{H}_\pi^{(k)}(E_\pi) \hat{\rho}^{(k)}(t, E_\pi; E) \\
&\qquad - e^{-\frac{1}{2} i\hbar \partial_t^{(\rho)} \partial_{E_\pi}^{(H)}} \hat{\rho}^{(k)}(t, E_\pi; E) \hat{H}_\pi^{(k)\dagger}(E_\pi)
\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}
&\left(i\hbar\partial_t \hat{\rho}^{(k)} \right)^{(-)}(t, E_\pi; E) \\
&= -\mathrm{tr}_{\rm ph} \int \frac{\mathrm{d}E'_\pi}{2\pi\hbar} 2\pi\hbar\delta\big(E'_\pi - E_\pi - \hat{H}_{\rm ph}\big) \\
&\times \Big[e^{\frac{1}{2} i\hbar \partial_t^{(\rho)} \partial_{E_\pi}^{(G)}} \hat{G}_\pi^{(k)}(E_\pi) \hat{F}^\dagger \hat{\rho}^{(k-1)}(t, E'_\pi; E) \hat{F} \\
&- e^{-\frac{1}{2} i\hbar \partial_t^{(\rho)} \partial_{E_\pi}^{(G)}} \hat{F}^\dagger \hat{\rho}^{(k-1)}(t, E'_\pi; E) \hat{F} \hat{G}_\pi^{(k)\dagger}(E_\pi) \Big].
\end{aligned} \tag{6.16}$$

Equations (6.11) and (6.13) (or equivalently, (6.14)) are the relevant result of this section. In $\partial_t \hat{\rho}^{(k)}$ we have distinguished two contributions. The first one, $(\partial_t \hat{\rho}^{(k)})^{(+)}$, is related to $\hat{\rho}^{(k)}$ itself and describes a pion (of class k) propagating with optical Hamiltonian $\hat{H}_\pi^{(k)}$, (cf. eq. (4.13)): it contains both, the "elastic" propagation of the pion inside the nucleus, and the quasi-elastic steps in which the pion of class k becomes of class $k+1$. In this sense it can be called the *annihilation* part of $\partial_t \hat{\rho}^{(k)}$. The second contribution, $(\partial_t \hat{\rho}^{(k)})^{(-)}$, accounts for the quasi-elastic steps of the form $k-1 \to k$. From the point of view of $\hat{\rho}^{(k)}$ it is the *creation* part of $\partial_t \hat{\rho}^{(k)}$. The Dirac delta in this term indicates that the ph's are on-shell. Mathematically, this is a direct consequence of having chosen all the ph times as $+\infty$ in the definition of the pionic density, (6.9).

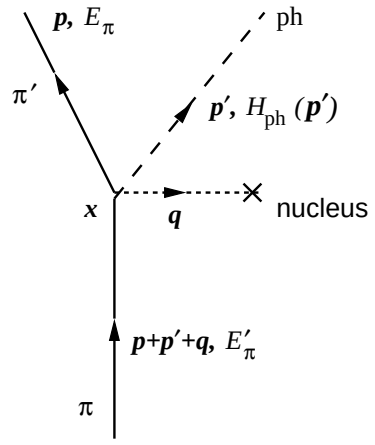

Fig. 4. Quasi-elastic process in a finite system.

Before studying eq. (6.14) in more detail, let us see how it works in a simple case:

$$\hat{F} = \int \mathrm{d}^3x g(\boldsymbol{x}) \hat{\phi}_{\rm ph}(\boldsymbol{x}) \hat{\phi}_\pi^\dagger(\boldsymbol{x}) \hat{\phi}_\pi(\boldsymbol{x}) \tag{6.17}$$

and $\hat{H}_0^{(\pi)}$, $\hat{H}_{\rm ph}$ functions of the momentum only. Let us first put the creation part of $\partial_t \hat{\rho}^{(k)}$ in Wigner's form. In a first step

$$\begin{aligned}
&\left[\mathrm{tr}_{\rm ph} \int \frac{\mathrm{d}E'_\pi}{2\pi\hbar} 2\pi\hbar\delta(E'_\pi - E_\pi - \hat{H}_{\rm ph}) \right. \\
&\qquad \left. \times \hat{F}^\dagger \hat{\rho}^{(k-1)}(t, E'_\pi; E) \hat{F} \right] (\boldsymbol{x}_\pi, \boldsymbol{p}_\pi) \\
&= \int \frac{\mathrm{d}^3p'}{(2\pi\hbar)^3} \frac{\mathrm{d}^3q}{(2\pi\hbar)^3} g^2(\boldsymbol{x}_\pi, \boldsymbol{q}) \rho^{(k-1)} \\
&\qquad \times (\boldsymbol{x}_\pi, \boldsymbol{p}_\pi + \boldsymbol{p}' + \boldsymbol{q}; t, E_\pi + H_{\rm ph}(\boldsymbol{p}'); E),
\end{aligned} \tag{6.18}$$

where

$$g^2(\boldsymbol{x}, \boldsymbol{q}) := \int \mathrm{d}^3y e^{-i\boldsymbol{y}\cdot\boldsymbol{q}/\hbar} g^* \left(\boldsymbol{x} + \frac{1}{2}\boldsymbol{y} \right) g \left(\boldsymbol{x} - \frac{1}{2}\boldsymbol{y} \right) \tag{6.19}$$

and $\boldsymbol{q}$ is the momentum transferred to the nucleus (fig. 4).

Next, we shall retain only the leading order in $\hbar$. In this way several simplifications take place in $(\partial_t \rho)^{(-)}$: i) the exponentials $\exp[\pm\frac{1}{2} i\hbar \partial_t^{(\rho)} \partial_{E_\pi}^{(G)}]$ are unity at leading order in $\hbar$; ii) the pion propagator becomes free, $\hat{G}_\pi(E) \to (E - \hat{H}_0^{(\pi)} + i\eta)^{-1}$, because each loop in its self-energy gives a $\hbar$ factor (eq. (4.16)); iii) in the classical limit the operators commute (cf. eq. (3.9)); and, iv) using the identity (3.5)

$$g^2(\boldsymbol{x}, \boldsymbol{q}) \underset{\hbar \to 0}{\to} (2\pi\hbar)^3 \delta(\boldsymbol{q}) g^2(\boldsymbol{x}). \tag{6.20}$$

After some algebra, the creation part can be written as

$$\Big(\partial_t\rho^{(k)}\Big)^{(-)}(\boldsymbol{x},\boldsymbol{p};t,E_\pi;E) = \int \frac{\mathrm{d}^3x'\,\mathrm{d}^3p'}{(2\pi\hbar)^3}\frac{\mathrm{d}E'_\pi}{2\pi\hbar}\tilde{Q}(\boldsymbol{x},\boldsymbol{p},E_\pi;\boldsymbol{x}',\boldsymbol{p}',E'_\pi) \times\rho^{(k-1)}(\boldsymbol{x}',\boldsymbol{p}';t,E'_\pi;E)+\mathcal{O}(\hbar) \tag{6.21}$$

with

$$\tilde{Q}(\boldsymbol{x},\boldsymbol{p},E_\pi;\boldsymbol{x}',\boldsymbol{p}',E'_\pi) = \frac{1}{\hbar^2}g^2(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{x}')2\pi\hbar\delta\Big(E_\pi-H_0^{(\pi)}(\boldsymbol{p})\Big) \times2\pi\hbar\delta\big(E'_\pi-E_\pi-H_{\rm ph}(\boldsymbol{p}'-\boldsymbol{p})\big). \tag{6.22}$$

We can see that the quasi-elastic probability in this approximation is positive, local ($\boldsymbol{x}=\boldsymbol{x}'$) and momentum is conserved ($\boldsymbol{q}=0$). The ph are on-shell ($E'_\pi-E_\pi=H_{\rm ph}(\boldsymbol{p}'-\boldsymbol{p})$) and moreover the outgoing pions are also created on-shell. As noted in the introduction the latter fact is a typical consequence of taking $\hbar\to0$, due to the uncertainty principle. Mathematically, the factor $2\pi\hbar\delta\big(E_\pi-H_0^{(\pi)}(\boldsymbol{p})\big)$ follows from the imaginary part of the free pion propagator $\hat{G}_\pi(E)-\hat{G}_\pi^\dagger(E)$ in (6.16). Summarizing, the probability of quasi-elastic $\tilde{Q}$ obtained at leading order in $\hbar$ coincides with the result that would follow from carrying out a standard nuclear-matter calculation ($g=g_c$ constant) in lowest order in perturbation theory, plus a local density prescription ($g_c\to g(x)$ at the end of the nuclear matter calculation) [24]. This latter fact is quite remarkable since usually the local density approximation is put in by hand, and here it follows naturally as a semiclassical approximation.

The annihilation part of $\partial_t\hat{\rho}^{(k)}$ comes from computing the pion optical potential up to one loop and keeping the free and the imaginary parts (lowest order in $\hbar$ in eq. (4.17)). This gives

$$\Big(\partial_t\rho^{(k)}\Big)^{(+)}(\boldsymbol{x},\boldsymbol{p};t,E_\pi;E) = \big\{H_0^{(\pi)}(\boldsymbol{p}),\rho^{(k)}(\boldsymbol{x},\boldsymbol{p};t,E_\pi;E)\big\}_{\rm P}, -\tilde{R}(\boldsymbol{x},\boldsymbol{p};E_\pi)\rho^{(k)}(\boldsymbol{x},\boldsymbol{p};t,E_\pi;E)+\mathcal{O}(\hbar), \tag{6.23}$$

where

$$\tilde{R}(\boldsymbol{x},\boldsymbol{p};E_\pi) = \frac{1}{\hbar^2}g^2(\boldsymbol{x})\int\frac{\mathrm{d}^3p'}{(2\pi\hbar)^3}2\pi\hbar\delta \times\Big(E_\pi-H_{\rm ph}(\boldsymbol{p}-\boldsymbol{p}')-H_0^{(\pi)}(\boldsymbol{p}')\Big). \tag{6.24}$$

The first term of the right-hand side of eq. (6.23) describes a classical free propagation of the pion. The second term indicates a reaction probability rate given by $\tilde{R}(\boldsymbol{x},\boldsymbol{p},E_\pi)$. Unitarity is verified since

$$\tilde{R}(\boldsymbol{x},\boldsymbol{p};E_\pi) = \int\frac{\mathrm{d}^3x'\,\mathrm{d}^3p'}{(2\pi\hbar)^3}\frac{\mathrm{d}E'_\pi}{2\pi\hbar} \times\tilde{Q}(\boldsymbol{x}',\boldsymbol{p}',E'_\pi;\boldsymbol{x},\boldsymbol{p},E_\pi). \tag{6.25}$$

Of course, the right-hand side of (6.21) should be set to 0 for the elastic channel, $k=0$, and similarly, the reaction term is not present in (6.23) when no ph remains in the nucleus, $k=A$. Otherwise, $\tilde{Q}$ and $\tilde{R}$ are independent of k and E in this approximation. Also note that because the pions are on-shell the E_π-dependence can be dropped and eqs. (6.21) and (6.23) have the form of eq. (1.3), except that the classical propagation has been extracted from the quasi-elastic rate Q.

Unfortunately, beyond leading order in $\hbar$, eq. (6.14) displays two undesirable features: first, the dependence of $\hat{\rho}^{(k)}$ on E_π, which implies that virtual as well as real pions coexist in $\hat{\rho}^{(k)}$. We deal with this problem in the next section. And second, the non-locality in time: in general $\partial_t\hat{\rho}^{(k)}$ will depend on the previous history of $\hat{\rho}^{(k)}$ and $\hat{\rho}^{(k-1)}$. The instantaneous equations can be obtained by a method similar to that used for eq. (4.17): $\partial_t\hat{\rho}^{(k)}$ is given as a function of $\partial_t^n\hat{\rho}^{(k)}$ and $\partial_t^{n'}\hat{\rho}^{(k-1)}$, $n,n'=0,1,\ldots$, each time derivative carrying a factor $\hbar$. Then, higher-order derivatives on the right-hand side can be eliminated in terms of lower ones. For instance, writing eq. (6.14) with an obvious schematic notation, we have for $k=1,2$

$$\partial_t\rho^{(2)} = \sum_{n=0}^{\infty}\hbar^n N_n^{(+)}\partial_t^n\rho^{(2)}+\sum_{n=0}^{\infty}\hbar^n N_n^{(-)}\partial_t^n\rho^{(1)}, \tag{6.26a}$$

$$\partial_t\rho^{(1)} = \sum_{n=0}^{\infty}\hbar^n {N'}_n^{(+)}\partial_t^n\rho^{(1)}+\sum_{n=0}^{\infty}\hbar^n {N'}_n^{(-)}\partial_t^n\rho^{(0)}. \tag{6.26b}$$

Then, to first order in $\hbar$,

$$\begin{aligned}\partial_t\rho^{(2)} = {}& \Big\{N_0^{(+)}+\hbar N_1^{(+)}N_0^{(+)}+\mathcal{O}(\hbar^2)\Big\}\rho^{(2)} \\ &+\Big\{N_0^{(-)}+\hbar N_1^{(-)}{N'}_0^{(+)}+\mathcal{O}(\hbar^2)\Big\}\rho^{(1)} \\ &+\Big\{\hbar N_1^{(-)}{N'}_0^{(-)}+\mathcal{O}(\hbar^2)\Big\}\rho^{(0)}.\end{aligned} \tag{6.27}$$

We observe that the equations can be written in an instantaneous form at the price of expressing $\partial_t\rho^{(k)}$, not only in terms of $\rho^{(k)}$ and $\rho^{(k-1)}$, but using all densities $\rho^{(k')}$ with $k'\le k$. We shall see that a similar phenomenon occurs when the E_π-dependence is removed form $\hat{\rho}^{(k)}$.

7 Integration of the virtual pions

In the previous section we have succeeded in writing an equation, eq. (6.14), for the temporal evolution of the density matrix of pions which have scattered k times, with only pionic degrees of freedom. However, it is not a fully satisfactory one due to its dependence on the pion energy, E_π. Consider, for instance, the evolution of a pion after its creation (*i.e.* after the scattering): two frequencies are relevant, first that characteristic of the source, represented by E_π, and second that of the free evolution, $H_0^{(\pi)}(\boldsymbol{p}_\pi)$. Due to interference, after traveling some wavelengths only on-shell pions, those with $E_\pi=H_0^{(\pi)}(\boldsymbol{p}_\pi)$, will survive, unless another scattering takes place. Such a collapse of the

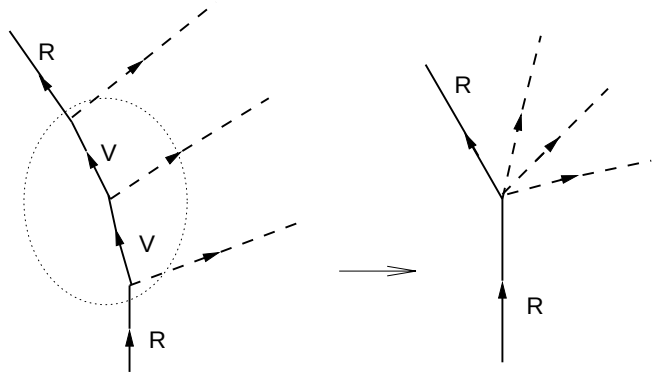

Fig. 5. Shrinking of virtual intermediate states (V) to produce an effective N-body quasi-elastic step between real pions (R).

wave function can also be regarded as a classical limit: if $\hbar$ were very small any time interval would be large compared with a few periods of the order of $\hbar/|E_\pi - H_0^{(\pi)}(\boldsymbol{p}_\pi)|$, provided that the particle is off-shell. Then, we expect the following relation to hold:

$$\hat{\rho}^{(k)}(t, E_\pi; E) \underset{\hbar\to 0}{\to} 2\pi\hbar\delta\Big(E_\pi - \hat{H}_0^{(\pi)}\Big) \tag{7.1}$$

up to a factor. This relation follows from the energy shell constraint eq. (4.21) and is checked in the next section (cf. eq. (8.13)).

Initially (in the incoming pion beam) the pion is on-shell. After several quasi-elastic steps the pion leaves the nucleus and is again on-shell since only on-shell pions can travel long distances to the detector. Between two successive collisions the pion can be (nearly) on-shell (*i.e.*, a *real* pion) or off-shell (a *virtual* pion). Consider a typical path of the pion through the nucleus, for instance RVVRVR (time increasing from right to left), where R stands for real and V for virtual pion, in this case with five quasi-elastic collisions. Each collision is of the type one-body mechanism, producing one ph. The virtual pions will travel a short distance and the Monte Carlo simulation only needs to trace the real ones. Within a classical limit, the real pions would follow a classical trajectory, whereas the path followed by the virtual pions shrinks. The situation is thus better described as a path RRR, that is, involving real pions only and two collisions. The first collision ejects two ph, a two-body quasi-elastic mechanism, whereas the second collision involves a three-body quasi-elastic mechanism (see fig. 5).

Advancing results of the next section, we can see the different properties of real and virtual pions as follows. Consider the classical limit directly in eq. (6.11). By neglecting the $\pm\omega_\pi/2$ (which carries an $\hbar$ upon Fourier transform to time representation) in $\hat{G}_\pi$ and going to time representation it turns out that $\hat{\rho}^{(k)}(t, E_\pi; E)$ is related to $\hat{\rho}^{(k-1)}(t, E'_\pi; E)$, *i.e.*, the number of pions of class k is proportional to the number of pions of class $k-1$ *at the same time*. This is only possible if they live during a very short period of time, and this is correct for virtual pions. Of course, for on-shell pions this would be an absurd consequence which is avoided because for them taking $\pm\omega_\pi/2 \to 0$ is not justified even in the classical limit: the quadratic $G_\pi G_\pi^\dagger$ divergence (at the on-shell point) is too strong. (This is also the reason for needing differential-like evolution equations, as (6.13) which have softer divergences than the integral-like (6.11).) The virtual pions are not only irrelevant to final cross-sections but their short-living times can only be achieved by interference which is very hard to reproduce by a simulation. Also, their existence implies that $\hat{\rho}(t, E_\pi; E)$ should be strongly non-positive definite.

Our goal is thus to integrate out the virtual pions and end up with an equation including only real pions. Because the real pions are on-shell their energy dependence is fixed and its corresponding density, to be denoted $\hat{f}_k(t, E)$, does not depend on E_π. In addition, in order to preserve unitarity at each time, and not only in the $t \to +\infty$ limit, we are led to the following definition for the density of on-shell pions:

$$\hat{f}_k(t, E) := \int \frac{\mathrm{d}E_\pi}{2\pi\hbar}\hat{\rho}^{(k)}(t, E_\pi; E), \quad k = 0, 1, 2, \ldots, \tag{7.2}$$

as the Monte Carlo density to work with. The suitability of this definition is further discussed over the end of this section. By construction it satisfies a conservation equation like eq. (6.12), namely,

$$\partial_t\left[\sum_{k=0}^{A} \mathrm{tr}\,\hat{f}_k(t, E)\right] = 0. \tag{7.3}$$

It is clear that the evolution equation for $\hat{\rho}^{(k)}(t, E_\pi; E)$ completely determines the evolution of $\hat{f}_k(t, E)$, however, whereas the equation for $\hat{\rho}^{(k)}$ involves pions of type k and $k-1$ (one-body mechanism) that for $\hat{f}_k$ will depend on pions of all classes $k' \le k$ (N-body quasi-elastic). In addition, initially $\hat{\rho}^{(0)}$ is determined by $\hat{f}_0$ (cf. (6.10)). So there is an autonomous equation for the densities $\hat{f}_k$. The question arises how to obtain such an equation. Simple integration over E_π in eqs. (6.11) or (6.13) does not work. Essentially the problem is how to invert eq. (7.2). To do so I make an *Ansatz* for the E_π distribution of the *real* pions in $\hat{\rho}^{(k)}(t, E_\pi; E)$:

$$\hat{\rho}^{(k)}(t, E_\pi; E) = \hat{\rho}_{\mathrm{R}}^{(k)}(t, E_\pi; E) + \hat{\rho}_{\mathrm{V}}^{(k)}(t, E_\pi; E), \tag{7.4}$$

where $\hat{\rho}_{\mathrm{R,V}}$ are the distributions of real and virtual pions in $\hat{\rho}$. The explicit form of these distributions is restricted by imposing the following conditions:

1) $\hat{\rho}_{\mathrm{R}}^{(k)}(t, E_\pi; E)$ should be constructed out of $\hat{f}_k(t, E)$, in order to be able to invert eq. (7.2).
2) Consistency with eq. (7.2) requires that

$$\hat{f}_k(t, E) = \int \frac{\mathrm{d}E_\pi}{2\pi\hbar}\hat{\rho}_{\mathrm{R}}^{(k)}(t, E_\pi; E), \tag{7.5a}$$

$$0 = \int \frac{\mathrm{d}E_\pi}{2\pi\hbar}\hat{\rho}_{\mathrm{V}}^{(k)}(t, E_\pi; E), \tag{7.5b}$$

because $\hat{f}_k$ is already the distribution of real pions without the redundant E_π-dependence.

3) $\hat\rho_{\rm R}^{(k)}, \hat\rho_{\rm V}^{(k)}$ should be Hermitian.
4) In the classical limit, $\hat\rho_{\rm V}^{(k)}$ should collapse to zero, due to quantal interference:

$$\hat\rho_{\rm V}^{(k)}(t,E_\pi;E) \underset{\hbar\to0}{\to} 0, \tag{7.6a}$$

$$\hat\rho_{\rm R}^{(k)}(t,E_\pi;E) \underset{\hbar\to0}{\to} 2\pi\hbar\delta\Big(E_\pi-\hat H_0^{(\pi)}\Big)\hat f_k(t,E). \tag{7.6b}$$

The δ-function in eq. (7.6b) follows from (7.1) and the factor $\hat f$ is obtained by normalization. In the classical limit, operators commute so there is no conflict between (7.6b) and the point 3) above.
5) For the elastic channel, we impose the constraint

$$\hat\rho_{\rm V}^{(k=0)} = 0. \tag{7.7}$$

This defines the elastic channel pions as real. Virtual pions only appear as a consequence of (hard) collisions, whereas the mean-field effects under the elastic evolution are regarded as soft. Technically this choice is needed to be able to close the equations below.

Let us see how such an *Ansatz* solves the problem. To alleviate the notation, let us write eqs. (6.11), (6.13) in the form

$$\hat\rho^{(k)} = M^{(k,k-1)}\hat\rho^{(k-1)}, \tag{7.8a}$$

$$\partial_t\hat\rho^{(k)} = N^{(k,k)}\hat\rho^{(k)} + N^{(k,k-1)}\hat\rho^{(k-1)}, \tag{7.8b}$$

or even more compactly

$$\rho = M\rho, \tag{7.9a}$$

$$\partial_t\rho = N\rho, \tag{7.9b}$$

where ρ is a (column) vector $(\hat\rho^{(0)},\hat\rho^{(1)},\dots,\hat\rho^{(A)})^T$, and M,N are matrices with respect to the index k and superoperators in Hilbert space (map operators onto operators). Also, eq. (7.2) can be written as

$$f = b\rho, \qquad b := \int\frac{{\rm d}E_\pi}{2\pi\hbar}\,. \tag{7.10}$$

In addition, $\rho_{\rm R}^{(k)}$ will be constructed out of f_k by some linear procedure $a^{(k)}$ to be specified later

$$\rho_{\rm R} = af = ab\rho, \tag{7.11a}$$

$$\rho_{\rm V} = (1-ab)\rho := P_{\rm V}\rho. \tag{7.11b}$$

$P_{\rm V}^{(k)}$ is the projector onto $\hat\rho_{\rm V}^{(k)}$. Equations (7.5) then read,

$$ba = 1, \qquad bP_{\rm V} = 0. \tag{7.12}$$

Note that b removes dependence on E_π, while a creates dependence on E_π, and M increases the index k by one unit. Equation (7.4) then implies

$$\rho = \rho_{\rm R}+\rho_{\rm V} = af + P_{\rm V}\rho = af + P_{\rm V}M\rho. \tag{7.13}$$

In the last step we have used the integral-like evolution equation for expressing the short-living virtual state in terms of its source. (In the elastic channel this is consistent due to (7.7).) This process can be iterated until the source is a real pion and in this way the virtual pions are integrated out. In practice this procedure can be carried out as follows. Using previous equations and the fact that ∂_t and b commute with each other, we can write the following chain of relations:

$$\begin{aligned}\partial_t f &= b\partial_t\rho = b\,N\rho\\ &= bN(af+P_{\rm V}M\rho)\\ &= bNaf + bNP_{\rm V}M(af+P_{\rm V}M\rho)\\ &= bNaf + bNP_{\rm V}Maf\\ &\quad + bNP_{\rm V}MP_{\rm V}M(af+P_{\rm V}M\rho)\\ &= \cdots .\end{aligned} \tag{7.14}$$

At each step, $\rho_{\rm V}$ is carried to smaller values of k since it appears with a new power of M. This downward recurrence ends due to (7.7) (the elastic-channel pions are real and no inconsistency arises). In this way all virtual pions can be eliminated and formally

$$\partial_t f = bN(1-P_{\rm V}M)^{-1}af. \tag{7.15}$$

This equation explicitly shows that f, as a vector, satisfies a closed equation. Let us expand eq. (7.15), with the obvious notation,

$$\begin{aligned}\partial_t\hat f_k \;=\; &\Big\{bN^{(k,k)}a^{(k)}\Big\}\hat f_k + \Big\{bN^{(k,k-1)}a^{(k-1)}\\ &+bN^{(k,k)}P_{\rm V}^{(k)}M^{(k,k-1)}a^{(k-1)}\Big\}\hat f_{k-1}\\ &+\Big\{bN^{(k,k-1)}P_{\rm V}^{(k-1)}M^{(k-1,k-2)}a^{(k-2)}\\ &+bN^{(k,k)}P_{\rm V}^{(k)}M^{(k,k-1)}P_{\rm V}^{(k-1)}\\ &\times M^{(k-1,k-2)}a^{(k-2)}\Big\}\hat f_{k-2}+\cdots\\ :=\;& R^{(k)}\hat f_k + Q^{(k,k-1)}\hat f_{k-1}\\ &+Q^{(k,k-2)}\hat f_{k-2}+\cdots,\end{aligned} \tag{7.16a}$$

$$\partial_t f \;=\; Rf+Qf, \tag{7.16b}$$

where R takes care of the propagation of the pion once it is produced and Q describes the quasi-elastic steps. (Note that in eq. (1.3) the elastic channel was included in Q while here it is given by $R^{(k=0)}$.)

Our actual procedure has been to remove any virtual pion by explicitly writing it in terms of the nearest real pion acting as a source for it. As a consequence only real pions do explicitly appear, and the hidden virtual intermediate states translate into effective N-body quasi-elastic probabilities, fig. 5.

In a classical limit, each factor $P_{\rm V}$ will increase the order in $\hbar$, due to eq. (7.6a), and only few terms will be needed in (7.16a). For instance, at leading order in $\hbar$ the one-body mechanism, $Q^{(k,k-1)}$, is dominant and the effective two-body quasi-elastic, $Q^{(k,k-2)}$, is the first quantum correction.

$R^{(k)}$ plays the role of an effective Hamiltonian for $\hat f_k$. Its actual form will depend on the concrete form of $a^{(k)}$,

which still has to be chosen, but using general properties of $a^{(k)}$ we still can say something about $R^{(k)}$

$$\begin{aligned} R^{(k)}\hat{f}_k(\omega_\pi, E) &= \Big(bN^{(k,k)}a^{(k)}f_k\Big)(\omega_\pi, E) \\ &= \frac{1}{i\hbar}\int\frac{\mathrm{d}E_\pi}{2\pi\hbar}\bigg(\hat{H}_\pi^{(k)}\Big(E_\pi+\frac{1}{2}\omega_\pi\Big)\hat{\rho}_{\mathrm{R}}^{(k)}(\omega_\pi, E_\pi; E) \\ &\quad -\hat{\rho}_{\mathrm{R}}^{(k)}(\omega_\pi, E_\pi; E)\hat{H}_\pi^{(k)\dagger}\Big(E_\pi-\frac{1}{2}\omega_\pi\Big)\bigg) \\ &= \frac{1}{i\hbar}\Big[\hat{H}_0^{(\pi)}, \hat{f}_k(\omega_\pi, E)\Big] \\ &\quad +\frac{1}{i\hbar}\int\frac{\mathrm{d}E_\pi}{2\pi\hbar}\bigg(\hat{V}_k\Big(E_\pi+\frac{1}{2}\omega_\pi\Big)\hat{\rho}_{\mathrm{R}}^{(k)}(\omega_\pi, E_\pi; E) \\ &\quad -\hat{\rho}_{\mathrm{R}}^{(k)}(\omega_\pi, E_\pi; E)\hat{V}_k^\dagger\Big(E_\pi-\frac{1}{2}\omega_\pi\Big)\bigg), \end{aligned} \tag{7.17}$$

where $\hat{V}_k = \hat{H}_\pi^{(k)} - \hat{H}_0^{(\pi)}$ is the optical potential. Further, for $\hbar \to 0$, using eq. (7.6b)

$$\begin{aligned} R^{(k)}\hat{f}_k(\omega_\pi, E) \approx \frac{1}{i\hbar}\bigg(&\hat{H}_\pi^{(k)}\Big(\bar{E}_\pi+\frac{1}{2}\omega_\pi\Big)\hat{f}_k(\omega_\pi, E) \\ &-\hat{f}_k(\omega_\pi, E)\hat{H}_\pi^{(k)\dagger}\Big(\bar{E}_\pi-\frac{1}{2}\omega_\pi\Big)\bigg), \end{aligned} \tag{7.18}$$

where $\bar{E}_\pi = \hat{H}_0^{(\pi)}$ is the on-shell energy.

In the special case $k = 0$, eq. (7.7) completely determines $a^{(k=0)}$ with the help of eq. (6.10)

$$a^{(k=0)} = 2\pi\hbar\delta(E - E_\pi), \tag{7.19}$$

which obviously satisfies all other requirements 1)–4), and the equation $\partial_t\hat{f}_{k=0} = R^{(k=0)}\hat{f}_{k=0}$ is nothing else than eq. (4.13).

In the general case, the super-operator $a^{(k)}$ can be chosen fairly arbitrarily. A natural choice is

$$\begin{aligned} \Big(a^{(k)}\hat{f}_k\Big)(t, E_\pi; E) &= i\hbar\hat{G}_\pi^{(k)}(E_\pi)\hat{f}_k(t, E) \\ &\quad -\hat{f}_k(t, E)i\hbar\hat{G}_\pi^{(k)\dagger}(E_\pi), \quad k \neq 0. \end{aligned} \tag{7.20}$$

Clearly, this choice satisfies the requirements 1) and 3), above. The point 2) follows from

$$i\hbar\hat{G}_\pi(E_\pi) = \int \mathrm{d}t\, e^{iE_\pi t/\hbar}\hat{U}_\pi(t)\theta(t), \tag{7.21}$$

where $\hat{U}_\pi(t)$ is the evolution operator, and unitarity, *i.e.* $\hat{U}_\pi(t = 0) = 1$. If $\hbar \to 0$, $\hat{G}_\pi$ and $\hat{f}$ commute and $\hat{G}_\pi$ approaches the free propagator,

$$i\hbar\hat{G}_\pi^{(k)}(E_\pi) - i\hbar\hat{G}_\pi^{(k)\dagger}(E_\pi) \to 2\pi\hbar\delta\Big(E_\pi - \hat{H}_0^{(\pi)}\Big), \tag{7.22}$$

which is the point 4), above. Other choices are possible, for instance taking only $\mathrm{Im}\,\hat{G}_\pi$, or taking $\hat{G}_\pi^{\mathrm{free}}$. Another possibility would be to use $E_\pi \pm \frac{1}{2}\omega_\pi$ instead of E_π in $\hat{G}_\pi$ in (7.20) (in frequency representation), however this choice turns out to be inappropriate for computing N-body absorption processes (cf. next section). As compared to that, our choice of $a^{(k)}$ is the *instantaneous* version since it relates $\rho_{\mathrm{R}}^{(k)}$ to f_k at the same time. It is noteworthy that the definition of $a^{(k)}$ has some resemblance with the formulas invoked in the standard cascade method [1,11].

Equations (7.16) are, perhaps, the main result of the paper. They describe the evolution of the purely pionic density matrix containing only real pions. They make explicit the N-body quasi-elastic rates $Q^{(k,k')}$ seen by the pion as it propagates through the nucleus. These rates are the input to be used in the cascade method. In appendix D these formulas are trivially extended to include absorption. They are analyzed in the next section.

It is noteworthy that the concrete choice of $a^{(k)}$ cannot affect the evolution of $\hat{f}_k$, if computed to all orders, that is, including all N-body mechanisms, since its definition (7.2) is independent of $a^{(k)}$. (The right-hand side of (7.15) is actually independent of a.) Obviously, different choices introduce different organizations of the series. This ambiguity is related to that in the separation into real and virtual pions. In this regard, it would be very interesting to choose $a^{(k)}$ so that not only $bP_{\mathrm{V}}^{(k)} = 0$, but also $bN^{(k,k)}P_{\mathrm{V}}^{(k)} = 0$. This essentially means that the real and virtual components of the density are separately preserved under *elastic* evolution, *i.e.* in the absence of collisions. In this case eqs. (7.16a) would simplify considerably. However, it is not clear how to impose this property or even whether it is consistent with the other requirements set on $a^{(k)}$.

We also note that the formulas (7.16) involve no approximations, except that of not including crossed graphs, that is, neglecting Bose symmetry of the ph's, as explained at the beginning of sect. 6[3].

In eq. (6.9) we introduced the density matrix $\hat{\rho}^{(k)}(t, E_\pi; E)$ for class k pions, depending on the time and energy of the pion and also on the conserved total energy E. It can be noted that in most formulas, including (6.11), (6.13), (7.2) and (7.20), E appears only as a parameter in the densities. An exception is (7.19), which refers to the elastic channel, and so a separated discussion is needed for that case.

For pions of class $k \geq 1$, E is just a parameter. We could as well introduce a new density $\hat{\rho}^{(k)}(t, E_\pi)$, without E-dependence, by integrating $\hat{\rho}^{(k)}(t, E_\pi; E)$ over E, and rederive all equations for it. If this is done, the density of *real* pions defined in (7.2) becomes $\hat{f}_k(t)$, which, recalling the "energy average" in (4.8), corresponds to the standard equal-time pionic density for pions of class k. In this light the integration of virtual pions is related to the closing of a set of equations for the equal-time densities. After the pion has scattered once or more, the initial energy E is no longer relevant and the corresponding N-body reaction

[3] Another simplification has been done in appendix D for absorption, this time just for obtaining a simpler presentation, namely, to disregard ph self-energy graphs.

probabilities do not depend on E, a fact explicitly verified in the next section.

On the other hand, for pions in the elastic channel E is quite relevant and coincides with E_π (cf. (6.10)). Integrating over E_π to obtain $\hat{f}_k(t,E)$ in (7.2) merely removes this redundant energy dependence. It should be noted that, being conserved, E is a known datum in the collision experiment, thus it does not seem advisable to remove this information by going further and integrate over E to work with the equal-time elastic-channel density matrix. Indeed, as verified in the next section, the N-body reaction probabilities $Q^{(N,0)}$ and $A^{(N,0)}$ display an explicit dependence on the pion energy.

As noted in sect. 4, when a space-time Wigner form is used (as opposed to an equal-time formulation) it is customary to use a transport and an energy shell constraint equation. In this work we have used the differential-like transport equation, (6.13), and the integral-like equation (6.11) as an equivalent set of equations. Actually the integral-like equation contains both the transport and the constraint equations (except for the elastic channel) and not surprisingly this equation is responsible for putting the particles on-shell as $\hbar$ goes to zero in the present approach (cf. eq. (8.12)).

8 N-body effective quasi-elastic and absorption mechanisms

In this section we shall work out the consequences of the previous scheme in the simplest cases. To simplify, we shall study only nuclear matter ($g(\boldsymbol{x}) = \text{constant} = g$ in (6.17), $K(\boldsymbol{x},\boldsymbol{y}) = \text{constant} = \kappa$ in (D.1), and $\rho_\pi(\boldsymbol{x},\boldsymbol{p};t,E)$ independent of $\boldsymbol{x}$), free propagators, locality in time, and in general lowest orders in $\hbar$. As was shown in sect. 6 all these assumptions are compatible with $\hbar \to 0$.

Let us study first the one-body quasi-elastic $Q^{(1,0)}$:

$$Q^{(1,0)} = bN^{(1,0)}a^{(0)} + bN^{(1,1)}P_{\rm V}^{(1)}M^{(1,0)}a^{(0)}. \tag{8.1}$$

In leading order of $\hbar$ only the first term contributes (besides in nuclear matter and using free Hamiltonians $N^{(k,k)} = 0$). The result can be obtained more directly from eqs. (6.21), (6.22):

$$Q^{(1,0)}(\boldsymbol{x},\boldsymbol{p};\boldsymbol{x}',\boldsymbol{p}';E) = \frac{1}{\hbar^2}g^2\delta(\boldsymbol{x}-\boldsymbol{x}') \times 2\pi\hbar\delta\Big(E - H_0^{(\pi)}(\boldsymbol{p}) - H_{\rm ph}(\boldsymbol{p}'-\boldsymbol{p})\Big). \tag{8.2}$$

The one-body absorption rate, $A^{(1,0)}$ is (the operation P plays a similar role in absorption as N does in quasi-elastic, see appendix D for details)

$$A^{(1,0)} = Pa^{(0)} \tag{8.3}$$

that can be worked out to give

$$A^{(1,0)}(\boldsymbol{x},\boldsymbol{p};E) = \frac{\kappa^2}{\hbar^2}2\pi\hbar\delta(E - H_{\rm ph}(\boldsymbol{p})). \tag{8.4}$$

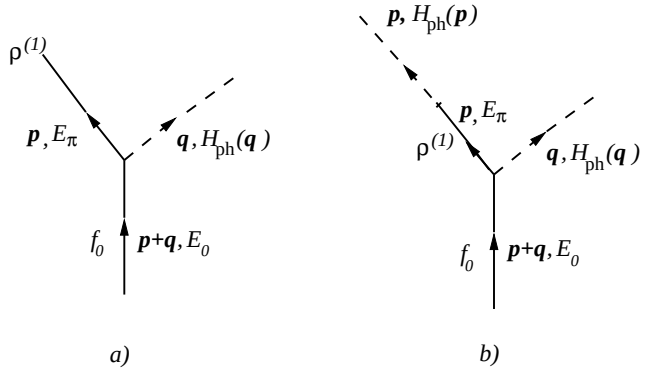

Fig. 6. a) Quasi-elastic step: the incoming pion and outgoing ph are on-shell. The outgoing pion can be real or virtual. b) Absorption process.

However, at leading order in $\hbar$, $E = H_0^{(\pi)}(\boldsymbol{p})$ which is incompatible with the δ-function in eq. (8.4) in physical cases: the pion and the ph cannot both be on-shell and as a consequence there is no one-body absorption in the classical limit.

In order to study more complicated cases, a drastic simplification in the notation is convenient. Consider the relation $\hat{\rho}^{(1)} = M^{(1,0)}a^{(0)}\hat{f}_0$ which is represented in fig. 6a. Using (6.11) for $M^{(1,0)}$ and (7.19) for $a^{(0)}$, it can be written as

$$\hat{\rho}^{(1)}(\omega_\pi,E_\pi;E) = \hat{G}_\pi^{(1)}\Big(E_\pi + \frac{1}{2}\omega_\pi\Big)\,{\rm tr}_{\rm ph} \times\big\{2\pi\hbar\delta(E - E_\pi - \hat{H}_{\rm ph})\hat{F}^\dagger\hat{f}_0(\omega_\pi,E)\hat{F}\big\} \times\hat{G}_\pi^{(1)\dagger}\Big(E_\pi - \frac{1}{2}\omega_\pi\Big). \tag{8.5}$$

In what follows, we will use E_0 to denote the total energy E and $f_0(t)$ to denote (the full Wigner's form of) $\hat{f}_0(t,E_0)$. In addition we introduce the notations (see fig. 6a)

$$\begin{aligned} E &:= E_\pi - H_0^{(\pi)}(\boldsymbol{p}),\\ \bar{E} &:= E_0 - H_{\rm ph}(\boldsymbol{q}) - H_0^{(\pi)}(\boldsymbol{p}),\\ \tilde{G}(E) &:= (E+i\eta)^{-1}. \end{aligned} \tag{8.6}$$

Furthermore, we will focus on the $\hbar$ and energy time dependence. Integration over momenta, vertex operators ($\hat{F}$, $\hat{K}$), traces, etc., will be implicit. Schematically, the relationship $\hat{\rho}^{(1)} = M^{(1,0)}a^{(0)}\hat{f}_0$ will look (in time representation, where the product of frequency functions becomes a convolution over times)

$$\rho^{(1)}(t,E) = \int {\rm d}\tau\hbar^2\big(\tilde{G}\tilde{G}^\dagger\big)(\tau,E) \times f_0(t-\tau)2\pi\hbar\delta(E-\bar{E}). \tag{8.7}$$

The total energy E_0 (called E in the previous sections) is not displayed. The operator $\big(\tilde{G}\tilde{G}^\dagger\big)(\tau,E)$ comes from

$\hat{G}_\pi(E_\pi + \frac{1}{2}\omega_\pi)\hat{G}^\dagger_\pi(E_\pi - \frac{1}{2}\omega_\pi)$, namely,

$$\hbar^2\big(\tilde{G}\tilde{G}^\dagger\big)(\tau,E) = \int \frac{\mathrm{d}\omega}{2\pi\hbar} e^{-i\omega\tau/\hbar}\hbar^2 \times\tilde{G}\bigg(E+\frac{1}{2}\omega\bigg)\tilde{G}^\dagger\bigg(E-\frac{1}{2}\omega\bigg). \quad (8.8)$$

This is the kernel of the super-operator M and controls how (real and virtual) pions of class k are produced out of pions of class $k-1$ and their subsequent propagation. In terms of a standard diagrammatic calculation (for instance in nuclear matter) this would correspond to compute the graphs associated to the transition amplitude between the initial and final states and then to square it to get the transition probability, or equivalently, to compute the self-energy graphs of the initial state and apply Cutkosky rules to associate each cut of the graph to a contribution to the transition probability to a concrete final state.

The exact integration over ω in (8.8) is easily performed, but an expansion in powers of $\hbar$ is more convenient for our purposes. To get such an expansion, we start with the identity

$$\hbar^2\partial_\tau\big(\tilde{G}\tilde{G}^\dagger\big)(\tau,E) = i\hbar\big(\tilde{G}-\tilde{G}^\dagger\big)(\tau,E). \quad (8.9)$$

The right-hand side is the kernel of N and is a softer distribution than $\big(\tilde{G}\tilde{G}^\dagger\big)$. Some algebra yields

$$\begin{aligned} i\hbar\big(\tilde{G}-\tilde{G}^\dagger\big)(\tau,E) &= -2\hbar\,\mathrm{Im}\left[\delta\bigg(\tau+\frac{1}{2}i\hbar\partial_E\bigg)\tilde{G}(E)\right] \\ &= 2\pi\hbar\delta(\tau)\delta(E) + \hbar^2\delta'(\tau)\bar{P}\frac{1}{E^2} \\ &\quad +\mathcal{O}\big(\hbar^3\big), \end{aligned} \quad (8.10)$$

where

$$\bar{P}\frac{1}{E^k} := \mathrm{Re}\,\big(\tilde{G}(E)\big)^k = \frac{(-1)^k}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}E^{k-1}}P\frac{1}{E} \quad (8.11)$$

is a renormalized principal value[4]. Integration over τ then yields

$$\begin{aligned} \hbar^2\big(\tilde{G}\tilde{G}^\dagger\big)(\tau,E) &= -2\hbar\,\mathrm{Im}\left[\theta\bigg(\tau+\frac{1}{2}i\hbar\partial_E\bigg)\tilde{G}(E)\right] \\ &= 2\pi\hbar\theta(\tau)\delta(E) + \hbar^2\delta(\tau)\bar{P}\frac{1}{E^2} \\ &\quad +\mathcal{O}\big(\hbar^3\big). \end{aligned} \quad (8.12)$$

Comparing (8.10) and (8.12), we can see that $i\hbar\big(\tilde{G}-\tilde{G}^\dagger\big)$, and so the super-operator N, is instantaneous at leading order in $\hbar$, whereas $\hbar^2\big(\tilde{G}\tilde{G}^\dagger\big)$, or M, is not. The non-instantaneous piece in M would have been missed if we had taken a formal classical limit neglecting the terms $\pm\frac{1}{2}\omega$ in $\hbar^2\big(\tilde{G}\tilde{G}^\dagger\big)$. This is incorrect because the divergence at the on-shell pole is not integrable. Correspondingly, as we will see subsequently, the non-instantaneous piece contributes only to real pions and not to virtual ones. Substituting in eq. (8.7),

$$\rho^{(1)}(t,E) = 2\pi\hbar\delta(E-\bar{E}) \times\bigg(2\pi\hbar\delta(\bar{E})\partial_t^{-1} + \hbar^2\frac{1}{\bar{E}^2} + \mathcal{O}(\hbar^3)\bigg)f_0(t), \quad (8.13)$$

where $\partial_t^{-1} = \int_{-\infty}^t \mathrm{d}t$, and the symbol $\bar{P}$ is implicit. We can see now what f_1, $\rho^{(1)}_{\mathrm{R}}$ and $\rho^{(1)}_{\mathrm{V}}$ look like. Recalling

$$b = \int \frac{\mathrm{d}E}{2\pi\hbar}\,, \quad (8.14a)$$

$$\begin{aligned} a^{(k)} &= i\hbar\big[\tilde{G}-\tilde{G}^\dagger\big]^{\mathrm{inst.}}(\tau,E) \\ &= 2\pi\hbar\delta(E)\delta(\tau), \quad k\neq 0, \end{aligned} \quad (8.14b)$$

where in $a^{(k)}$ only the instantaneous part should be taken due to the missing $\pm\omega_\pi/2$ in (7.20), we have (presently $a^{(k)}$ acts by convolution over τ)

$$f_1(t) = \bigg(2\pi\hbar\delta(\bar{E})\partial_t^{-1} + \hbar^2\frac{1}{\bar{E}^2} + \mathcal{O}\big(\hbar^3\big)\bigg)f_0(t), \quad (8.15a)$$

$$\rho^{(1)}_{\mathrm{R}}(t,E) = 2\pi\hbar\delta(E)\bigg(2\pi\hbar\delta(\bar{E})\partial_t^{-1} + \hbar^2\frac{1}{\bar{E}^2} + \mathcal{O}(\hbar^3)\bigg)f_0(t), \quad (8.15b)$$

$$\begin{aligned} \rho^{(1)}_{\mathrm{V}}(t,E) &= 2\pi\hbar^3\frac{1}{\bar{E}^2}\big(\delta(E-\bar{E})-\delta(E)\big)f_0(t) \\ &\quad +\mathcal{O}(\hbar^4). \end{aligned} \quad (8.15c)$$

We can see that in $\rho^{(1)}_{\mathrm{R}}(t)$ the pions are on-shell, $E=0$, and also that it contains a piece, that with $\partial_t^{-1}f_0$, which depends on the whole previous history of the incoming pions, *i.e.*, $\rho^{(1)}_{\mathrm{R}}(t)$ contains long-lasting components indicating that the real pion at t could be produced from a quasi-elastic step which occurred some time ago. On the other hand, $\rho^{(1)}_{\mathrm{V}}$ does not contain such components; the number of virtual pions depends on the instantaneous number of incoming pions indicating that the virtual pions are short-living states.

Now we can easily compute absorption and quasi-elastic rates. $Q^{(1,0)}$ will follow from

$$\begin{aligned} Q^{(1,0)}f_0(t) &= \int \frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{d}\tau i\hbar\big(\tilde{G}-\tilde{G}^\dagger\big)(\tau,E) \\ &\quad \times 2\pi\hbar\delta(E-\bar{E})f_0(t-\tau) \\ &= \big(2\pi\hbar\delta(\bar{E}) + \mathcal{O}\big(\hbar^2\big)\big)f_0(t), \end{aligned} \quad (8.16)$$

[4] As a technical remark, note that for fairly general test function spaces, the principal value of $1/x$ is well defined as a distribution, thus, unlike $P(1/x^k)$, the construction $\bar{P}(1/x^k)$ is also a well-defined distribution for $k=2,3,\ldots$ (finite when applied to test functions) since the derivative of a distribution is again a distribution, defined through by-parts integration [60,61].

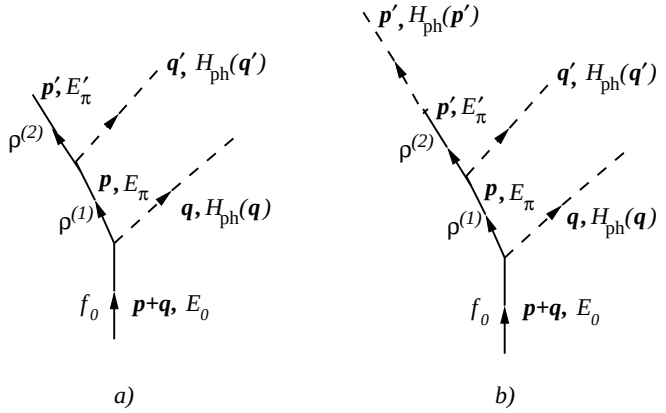

Fig. 7. a) Two-body quasi-elastic. b) Three-body absorption.

which coincides with (8.2). Analogously, $A^{(2,1)}$ and $A^{(2,0)}$

$$A^{(2,1)}f_1 = Pa^{(1)}f_1 = \int \frac{\mathrm{d}E}{2\pi\hbar} \times 2\pi\hbar\delta(E-E_A)\rho_{\mathrm{R}}^{(1)}(t,E), \tag{8.17a}$$

$$A^{(2,0)}f_0 = P\rho_{\mathrm{V}}^{(1)} = \int \frac{\mathrm{d}E}{2\pi\hbar} \times 2\pi\hbar\delta(E-E_A)\rho_{\mathrm{V}}^{(1)}(t,E), \tag{8.17b}$$

where $E_A := H_{\mathrm{ph}}(\boldsymbol{p}) - H_0^{(\pi)}(\boldsymbol{p})$ (see fig. 6b). Using the expressions in (8.15) and the fact that $E_A \neq 0$, as discussed for $A^{(1,0)}$, we get

$$A^{(2,1)} = 0, \tag{8.18a}$$

$$A^{(2,0)}f_0(t) = 2\pi\hbar^3\frac{1}{\bar{E}^2}\delta(E_A-\bar{E})f_0(t) + \mathcal{O}(\hbar^5). \tag{8.18b}$$

$Q^{(1,0)}$ and $A^{(2,0)}$ are then the lowest-order quasi-elastic and absorption mechanisms. Both are positive definite ($\bar{P}\frac{1}{\bar{E}^2}$ is positive outside the pole)[5].

To display further features of the present scheme we shall study the coefficients $Q^{(2,0)}$ and $A^{(3,0)}$. Again schematically

$$Q^{(2,1)}f_1(t) = \int \frac{\mathrm{d}E'}{2\pi\hbar}\,\mathrm{d}\tau'\frac{\mathrm{d}E}{2\pi\hbar}i\hbar\Big(\tilde{G}-\tilde{G}^\dagger\Big)(\tau',E') \times 2\pi\hbar\delta(E-E'+\bar{E}')\rho_{\mathrm{R}}^{(1)}(t-\tau',E) = \big(2\pi\hbar\delta(\bar{E}') + \mathcal{O}\big(\hbar^2\big)\big)f_1(t), \tag{8.19}$$

where $E' = E'_\pi - H_0^{(\pi)}(\boldsymbol{p}')$, $\bar{E}' = H_0^{(\pi)}(\boldsymbol{p}) - H_0^{(\pi)}(\boldsymbol{p}') - H_{\mathrm{ph}}(\boldsymbol{q}')$ (see fig. 7a).

[5] Using $E_\pi \pm \frac{1}{2}\omega_\pi$ as arguments in the pion propagator in eq. (7.20) (in frequency representation), would introduce further terms in the definition of $a^{(k)}$, (as in (8.10)) compared to (8.14b)). With this alternative definition $\rho_{\mathrm{R}}^{(1)}(t,E)$ would no longer be proportional to $\delta(E)$ and, as a consequence, $A^{(2,1)}$ would not vanish and $A^{(2,0)}$ would not be positive at leading order (their sum, however, would not change).

$Q^{(2,0)}$ is obtained by taking $\rho_{\mathrm{V}}^{(1)}$ instead of $\rho_{\mathrm{R}}^{(1)}$, in (8.19):

$$Q^{(2,0)}f_0(t) = 2\pi\hbar^3\frac{1}{\bar{E}^2}\big(\delta(\bar{E}'+\bar{E}) - \delta(\bar{E}')\big)f_0(t) + \mathcal{O}(\hbar^4). \tag{8.20}$$

Note first that $Q^{(2,1)} = Q^{(1,0)}$, at least at leading order. In general $Q^{(k+s,s)} = Q^{(k,0)}$ would be most desirable, but given the asymmetry in the definitions of $a^{(k\neq 0)}$, and $a^{(0)}$ it might not be true for higher orders in $\hbar$. Secondly, note that $Q^{(2,0)}$ is not positive definite. This is a consequence of eq. (7.5b), and it was expected because it is the first quantum correction to the quasi-elastic. Indeed

$$Q^{(2,1)}f_1(t) = 2\pi\hbar\delta(\bar{E}')2\pi\hbar\delta(\bar{E})\partial_t^{-1}f_0(t) + \mathcal{O}(\hbar^3), \tag{8.21}$$

which shows that effectively $Q^{(2,0)} \approx \hbar Q^{(2,1)}$. It is also interesting that as $\bar{E} \to 0$ in (8.20) and the intermediate pion (line $\boldsymbol{p}$ in fig. 7a) approaches its mass-shell, the two δ-functions cancel: this is the effect of subtracting real pions from $\rho^{(1)}$. Then, even if the $\bar{P}\frac{1}{\bar{E}^2}$ distribution were not already renormalized and finite[6] the subtraction $\rho^{(1)} - \rho_{\mathrm{R}}^{(1)}$ will produce a finite result.

Similarly for absorption:

$$A^{(3,1)}f_1 = \int \frac{\mathrm{d}E'}{2\pi\hbar}2\pi\hbar\delta(E'-E'_A)\rho_{\mathrm{VR}}^{(2)}, \tag{8.22a}$$

$$A^{(3,0)}f_0 = \int \frac{\mathrm{d}E'}{2\pi\hbar}2\pi\hbar\delta(E'-E'_A)\rho_{\mathrm{VV}}^{(2)}, \tag{8.22b}$$

with $E'_A = H_{\mathrm{ph}}(\boldsymbol{p}') - H_0^{(\pi)}(\boldsymbol{p}') \neq 0$ (see fig. 7b), and the result

$$A^{(3,1)}f_1(t) = 2\pi\hbar^3\frac{1}{\bar{E}'^2}\delta(E'_A-\bar{E}')f_1(t) + \mathcal{O}(\hbar^4), \tag{8.23a}$$

$$A^{(3,0)}f_0(t) = 2\pi\hbar\frac{\hbar^2}{E'^2_A}\frac{\hbar^2}{\bar{E}^2}\Big(\delta(E'_A-\bar{E}'-\bar{E}) - \delta(E'_A-\bar{E}')\Big)f_0(t) + \mathcal{O}(\hbar^7). \tag{8.23b}$$

Once again $A^{(3,1)} = A^{(2,0)}$, also $A^{(3,0)}$ is not positive and the δ's cancel as $\bar{E} \to 0$. Furthermore, in both cases $Q^{(2,0)}$ and $A^{(3,0)}$ average to zero (as functions of $\bar{E}'$), again a direct consequence of (7.5b). Although after momentum integration $Q^{(2,0)}$ and $A^{(3,0)}$ will not be zero, they will be small. In our formalism this is reflected in the fact that both have higher-order powers of $\hbar$. Remember that $\hbar \to 0$ should be understood as a physical limit: it is not $\hbar$ that is small, rather the coefficients with higher orders of $\hbar$ are smaller if the classical limit applies.

Even though $Q^{(2,0)}$ and $A^{(3,0)}$ involve the propagation of intermediate (virtual) pions, both rates are instantaneous in leading order implying that such states last a short time. A more detailed treatment equally shows that they run a short distance, *i.e.*, Q and A are local at leading order.

[6] This is the case when Cutkosky rules [37] are directly applied in nuclear matter.

9 Summary and conclusions

In the previous sections we have reduced a complex many-body evolution equation to something more similar to the master equation of the cascade approach (1.3). The task has been that of rewriting the Schrödinger equation and, more importantly, that of removing the unwanted degrees of freedom, namely, the ph and the virtual-pion degrees of freedom, which are not present in the cascade-like calculation of ref. [24]. We have then obtained a set of equations for the pionic matrix density, eqs. (7.16), in which a classical expansion can be made naturally.

We have started by formulating the relation between the S-matrix and cross-section in the phase space approach, (2.13), then we have shown that the analogous relation can be written using directly the evolution operator, (2.21). This relation is closer to a cascade model since the simulation is set just to provide the transition probability from an initial point in phase space to any other point at large later times. Both in quantum-mechanical calculations and in cascade methods, the transition probability (evolution operator) is obtained by solving the associated differential equation (Schrödinger equation). The rest of the paper is devoted to compute the input to be used for that equation. For this input to be useful it has to be written in a way that connects with cascade calculations, and we have argued that this suggests to carry out an expansion around the classical limit. We emphasize, however, that the validity of the final formulas (6.14) and (7.16) do not rely on classical-like approximations (they are not leading-order terms of a classical expansion). They are translations of fully quantum-mechanical relations, reformulated in a phase space form with the help of the Wigner transformation. (Nevertheless, we recall that these formulas do not implement indistinguishability of the ph and this can be regarded as a classical-like feature.)

In sects. 3 and 4 we have introduced the necessary formalism related to the Wigner formulation. The phase space (space momentum) part of this formulation has been exploited already by various authors in the literature even in the specific subject of quantum scattering [38–41]. The relevance of the *time energy* form of the Wigner transformation for many-body problems is noted in sect. 4 for the elastic channel and in sect. 5 for inelastic channels (within a simplified model). This form is used ubiquitously later in the paper. In sects. 6 and 7 we have introduced the necessary definitions of the matrix densities until we have pinned down the quantity, $\hat{f}$ in (7.2), that can naturally be identified with the density of particles described by a cascade-like method and we have also derived the evolution equations satisfied by this quantity, (7.15). (We note that the validity of these evolution equations holds regardless of the interpretation given to $\hat{f}$.) At the same time, we have systematically studied the classical limit of the main formulas to verify their consistency and intuitive meaning, besides, is also within a classical expansion where they can be simple enough to be of any utility. The classical-like expansion of (7.16) has been pursued further in sect. 8 to isolate the leading contributions to the N-body quasi-elastic and absorption mechanisms.

Among the conclusions of this study we note:

1. The separation of the pion "width" in the nucleus into a quasi-elastic one plus an absorptive one, is achieved by means of the Cutkosky rules [24,53] in the diagrammatic approach. Here, we have already been working with the imaginary parts of the propagators from the very beginning and with its different analytical cuts, so that no further separation is needed.

2. We have been able to give a meaning to the concept of effective N-body quasi-elastic and absorption probabilities. (However, beyond lowest orders in $\hbar$ the prescription is not unique: different choices of $a^{(k)}$ in eq. (7.20) could have been taken.) In particular, an explicit answer is given to the problem of distinguishing three-body absorption from a quasi-elastic step followed by two-body absorption. The "genuine" three-body absorption is given by $A^{(3,0)}$ in (8.23b), whereas the quasi-elastic one followed by two-body absorption is described by $Q^{(1,0)}$ in (8.16) and $A^{(3,1)}$ in (8.23a). The rule to obtain $A^{(3,0)}$ is essentially to subtract $A^{(3,1)}$ from the full calculation of $\pi \to (\mathrm{ph})^3$ obtained through a proper Feynman diagrammatic calculation. In practice this is done by computing the relevant pion self-energy graphs and then applying Cutkosky rules to pick up the imaginary part, corresponding to putting the final particles on their mass shell. The procedure of separation is further discussed in greater detail in [53].

3. Higher-order effective quasi-elastic and absorption probabilities are quantum corrections to lowest orders and they are not positive definite. This is a direct consequence of unitarity (conservation of number of thrown pions minus absorbed pions). A weaker condition, namely, unitarity for large times only, would be enough but it would require quantum interference which is prohibitive in a simulation. It is important to note that in physical cases the *genuine* three-body absorption is by far dominated by the collision of the pion with three nucleons which exchange heavier mesons, rather than by exchanging far off-shell pions. As this heavy mesons are necessarily virtual, the subtractions discussed here have no effect and their contribution is positive definite, very much the same as in the two-body absorption $A^{(2,0)}$ of our model [53]. As a matter of principle, the problem of negative probabilities can be handled by the known method of assigning weights to the particles as they cascade, in this case a negative weight. Unfortunately, this method introduces large statistical fluctuations. (This is the ubiquitous negative-sign problem in quantum simulations [54–56]) As noted, the concrete choice of $a^{(k)}$ in (7.15) cannot change the result if computed to all orders (although it may affect the rate of convergence of the expansion). Perhaps this ambiguity can be used for reducing the importance of the negative regions in the higher-order quasi-elastic and absorptions rates.

4. Nuclear matter and the local density prescription appear naturally in this scheme. Because the effective reaction rates comes about by integrating virtual degrees of freedom, which are quasi-local, the nuclear-matter calcula-

tion results as the leading order in a semi-classical expansion. However, one of the results of the study in ref. [24] is that it is important to take into account the finite range of the pion-nucleon interaction which is p-wave. This appears in our scheme only at higher than leading terms in $\hbar$.

5. Indistinguishability of particles has not been implemented in this scheme. This may or may not be more than a technical problem. It is certainly a quantum effect. (See [57] for the use of the Wigner transformation for an optimal treatment of the Bose symmetry of pions.) We have chosen to remove the ph's and follow only the pions as they cascade through the nucleus. It should be possible to do a treatment without this removal, that is, with explicit ph's (of course, the *virtual* ph's still have to be integrated out). Perhaps this could have some incidence on the problem related to the missing crossed graphs. In the context transport equations, the problem of antisymmetry of nucleons has been successfully addressed in extensions of quantum-molecular dynamics [58,59].

6. Although with pion-nucleus scattering in mind, the scheme is more general and could be of interest for other problems currently dealt with by Monte Carlo simulation methods. Furthermore, the scheme is exemplified with a simple model, but the final equations involve Green's functions which exist in any quantum many-body theory.

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Appendix A.

Let us prove eq. (2.10) for any free Hamiltonian of the form $\hat{H}_0 = \hat{H}_0(\boldsymbol{p})$. From the definition of $\hat{S}$ we have $[\hat{H}_0, \hat{S}] = 0$, then for any $\hat{\rho}$

$$e^{-it\hat{H}_0/\hbar}\hat{S}\hat{\rho}\hat{S}^\dagger e^{it\hat{H}_0/\hbar} = \hat{S}\, e^{-it\hat{H}_0/\hbar}\hat{\rho}\, e^{it\hat{H}_0/\hbar}\hat{S}^\dagger. \quad (A.1)$$

In Wigner's form, it implies

$$\int \frac{\mathrm{d}^3q\,\mathrm{d}^3y}{(2\pi\hbar)^3} e^{i\boldsymbol{q}(\boldsymbol{x}'-\boldsymbol{y})/\hbar - it\Delta H(\boldsymbol{q},\boldsymbol{p}')/\hbar} S(\boldsymbol{y},\boldsymbol{p}';\boldsymbol{x},\boldsymbol{p}) = \int \frac{\mathrm{d}^3q\,\mathrm{d}^3y}{(2\pi\hbar)^3} S(\boldsymbol{x}',\boldsymbol{p}';\boldsymbol{y},\boldsymbol{p}) e^{i\boldsymbol{q}(\boldsymbol{y}-\boldsymbol{x})/\hbar - it\Delta H(\boldsymbol{q},\boldsymbol{p})/\hbar} \quad (A.2)$$

with $\Delta H(\boldsymbol{q},\boldsymbol{p}) = H_0(\boldsymbol{p}+\frac{1}{2}\boldsymbol{q}) - H_0(\boldsymbol{p}-\frac{1}{2}\boldsymbol{q})$. Upon integration over $\boldsymbol{x}'$ and $\boldsymbol{b}$, we have

$$\int \mathrm{d}^2b\,\mathrm{d}^3x' S(\boldsymbol{x}',\boldsymbol{p}';\boldsymbol{b}+\boldsymbol{x}_\parallel,\boldsymbol{p}) = \int \mathrm{d}^3x'\,\mathrm{d}^2b \frac{\mathrm{d}q_\parallel\,\mathrm{d}y_\parallel}{2\pi\hbar} S(\boldsymbol{x}',\boldsymbol{p}';\boldsymbol{b}+\boldsymbol{y}_\parallel,\boldsymbol{p}) \times e^{i\boldsymbol{q}_\parallel(\boldsymbol{y}_\parallel-\boldsymbol{x}_\parallel)/\hbar - it\Delta H(\boldsymbol{q}_\parallel,\boldsymbol{p})/\hbar} \quad (A.3)$$

for large t only small values of ΔH can survive, and ΔH can be approximated by $\boldsymbol{v}(\boldsymbol{p})\cdot\boldsymbol{q}_\parallel$ with $\boldsymbol{v}$ the classical velocity. Then the $\boldsymbol{q}_\parallel$ integration gives $\boldsymbol{y}_\parallel = \boldsymbol{x}_\parallel + \boldsymbol{v}t$. But the left-hand side does not depend on t, then it cannot depend on $\boldsymbol{x}_\parallel$ either (unless $\boldsymbol{v}=0$, for which case there is no scattering at all). More technically, applying $\int \mathrm{d}t e^{i\omega t}\,\mathrm{d}x_\parallel e^{-i\boldsymbol{q}'_\parallel\cdot\boldsymbol{x}_\parallel}$ on both sides of (A.3), we have

$$0 = \big\{\delta(\omega) - \delta\big(\omega - \Delta H(\boldsymbol{q}_\parallel,\boldsymbol{p})\big)\big\} \int \mathrm{d}x_\parallel\, e^{-i\boldsymbol{q}_\parallel\cdot\boldsymbol{x}_\parallel} \times \int \mathrm{d}^2b\,\mathrm{d}^3x' S\big(\boldsymbol{x}',\boldsymbol{p}';\boldsymbol{b}+\boldsymbol{x}_\parallel,\boldsymbol{p}\big), \quad (A.4)$$

which implies (2.10) for $\boldsymbol{p}\neq 0$.

Appendix B.

We want to prove eqs. (6.5) and (6.6). By iterating eq. (6.6) towards lower k, the state $|E_\pi, E_1, \cdots, E_k\rangle_k$ must be given by

$$\begin{aligned}|E_\pi, E_1, \ldots, E_k\rangle_k &= \left[i\hbar\hat{G}_\pi^{(k)}(E_\pi) i\hbar\hat{G}_{\mathrm{ph}}^{(k)}(E_k)\frac{1}{i\hbar}\hat{F}^\dagger\right] \times\left[i\hbar\hat{G}_\pi^{(k-1)}(E_\pi+E_k) i\hbar\hat{G}_{\mathrm{ph}}^{(k-1)}(E_{k-1})\frac{1}{i\hbar}\hat{F}^\dagger\right]\cdots \cdots\left[i\hbar\hat{G}_\pi^{(1)}(E_\pi+E_k+\cdots+E_2) i\hbar\hat{G}_{\mathrm{ph}}^{(1)}(E_1)\frac{1}{i\hbar}\hat{F}^\dagger\right] \times|E_\pi+E_1+\cdots+E_k\rangle_0 \quad (B.1)\end{aligned}$$

and for $|E\rangle_0$ we choose the state without ph particles, $|0,E\rangle$. To illustrate the method it will be sufficient to prove eq. (6.5) just for $k=2$:

$$\begin{aligned}|E_\pi, E_1, E_2\rangle_2 &= [i\hbar\hat{G}_\pi^{(2)}(E_\pi) i\hbar\hat{G}_{\mathrm{ph}}^{(2)}(E_2)\frac{1}{i\hbar}\hat{F}^\dagger] \times\left[i\hbar\hat{G}_\pi^{(1)}(E_\pi{+}E_2) i\hbar\hat{G}_{\mathrm{ph}}^{(1)}(E_1)\frac{1}{i\hbar}\hat{F}^\dagger\right] \times|E_\pi+E_1+E_2\rangle_0 \quad (B.2)\\ |2,E\rangle' &= \int \frac{\mathrm{d}E_\pi}{2\pi\hbar}\frac{\mathrm{d}E_1}{2\pi\hbar}\frac{\mathrm{d}E_2}{2\pi\hbar} 2\pi\hbar\delta \times(E-E_\pi-E_1-E_2)|E_\pi,E_1,E_2\rangle_2 \\ &= \left\{\left[i\hbar\hat{G}_\pi^{(2)}\circ i\hbar\hat{G}_{\mathrm{ph}}^{(2)}\frac{1}{i\hbar}\hat{F}^\dagger\right] i\hbar\hat{G}_\pi^{(1)}\right\} \circ i\hbar\hat{G}_{\mathrm{ph}}^{(1)}\frac{1}{i\hbar}\hat{F}^\dagger|E\rangle_0. \quad (B.3)\end{aligned}$$

Using the analytic properties of the propagator, we find

$$\left[i\hbar\hat{G}_\pi^{(k)}\circ i\hbar\hat{G}_{\mathrm{ph}}^{(\ell)}\right](E) = i\hbar\hat{G}_\pi^{(k)}\left(E-\hat{H}_{\mathrm{ph}}^{(\ell)}\right), \quad k\geq\ell, \quad (B.4)$$

where $\hat{H}_{\rm ph}^{(\ell)}$ is the free ph Hamiltonian in the ℓ-th ph subspace. And similarly

$$\begin{aligned}&\left(\left\{\left[i\hbar\hat{G}_\pi^{(2)}\circ i\hbar\hat{G}_{\rm ph}^{(2)}\right]i\hbar\hat{G}_\pi^{(1)}\right\}\circ i\hbar\hat{G}_{\rm ph}^{(1)}\right)(E)\\&=i\hbar\hat{G}_\pi^{(2)}\left(E-\hat{H}_{\rm ph}^{(1)}-\hat{H}_{\rm ph}^{(2)}\right)i\hbar\hat{G}_\pi^{(1)}\left(E-\hat{H}_{\rm ph}^{(1)}\right)\\&=\left[i\hbar\hat{G}_\pi^{(2)}\circ i\hbar\hat{G}_{\rm ph}^{(1)}\circ i\hbar\hat{G}_{\rm ph}^{(2)}\right]\left[i\hbar\hat{G}_\pi^{(1)}\circ i\hbar\hat{G}_{\rm ph}^{(1)}\right](E),\end{aligned}\tag{B.5}$$

where we have used the fact that $\hat{G}_\pi^{(k)}$ do not contain operators related to the k'-th ph if $k'\leq k$. This is due to our assumption that only direct graphs are included in the propagators. The same steps in (B.5) go through when the $\hat{F}^\dagger$ operators are in place, as in eq. (B.3), because the last $\hat{F}^\dagger$ operator and $\hat{G}_{\rm ph}^{(1)}$ commute: our assumption is that each time $\hat{F}^\dagger$ creates the latter ph, in this case that labeled with (2). Then,

$$\begin{aligned}|2,E\rangle'=&\left[i\hbar\hat{G}_\pi^{(2)}\circ i\hbar\hat{G}_{\rm ph}^{(1)}\circ i\hbar\hat{G}_{\rm ph}^{(2)}\right]\frac{1}{i\hbar}\hat{F}^\dagger\\&\times\left[i\hbar\hat{G}_\pi^{(1)}\circ i\hbar\hat{G}_{\rm ph}^{(1)}\right]\frac{1}{i\hbar}\hat{F}^\dagger|0,E\rangle\end{aligned}\tag{B.6}$$

using eq. (6.1),

$$|2,E\rangle'=\hat{G}_2(E)\hat{F}^\dagger\hat{G}_1(E)\hat{F}^\dagger|0,E\rangle\tag{B.7}$$

and finally using eq. (5.9)

$$|2,E\rangle'=\hat{G}_2(E)\hat{F}^\dagger|1,E\rangle=|2,E\rangle,\tag{B.8}$$

in agreement with eq. (6.5).

Appendix C.

In order to prove eq. (6.11), let us apply the defining operator in eq. (6.9) on the recurrence (6.8):

$$\begin{aligned}&\hat{\rho}^{(k)}(\omega_\pi,E_\pi;E)\\&\quad=\hbar^2\hat{G}_\pi^{(k)}\left(E_\pi+\frac{1}{2}\omega_\pi\right)\int\frac{{\rm d}E_k}{2\pi\hbar}\,{\rm tr}_{\rm ph}\lim_{t_k\to+\infty}\\&\qquad\times\int\frac{{\rm d}\omega_k}{2\pi\hbar}e^{-i\omega_k t_k/\hbar}\hat{G}_{\rm ph}\left(E_k+\frac{1}{2}\omega_k\right)\\&\qquad\times\hat{F}^\dagger\hat{\rho}^{(k-1)}(\omega_\pi+\omega_k,E_\pi+E_k;E)\\&\qquad\times\hat{F}\hat{G}_{\rm ph}^\dagger\left(E_k-\frac{1}{2}\omega_k\right)\hat{G}_\pi^{(k)\dagger}\left(E_\pi-\frac{1}{2}\omega_\pi\right).\end{aligned}\tag{C.1}$$

It is convenient to do a Fourier transformation of ω_π in order to make explicit the dependence on ω_k:

$$\begin{aligned}&\hat{\rho}^{(k)}(t_\pi,E_\pi;E)\\&\quad=\int\frac{{\rm d}\omega_\pi}{2\pi\hbar}e^{-i\omega_\pi t_\pi/\hbar}\hbar^2\hat{G}_\pi^{(k)}\left(E_\pi+\frac{1}{2}\omega_\pi\right)\\&\qquad\times\int\frac{{\rm d}E_k}{2\pi\hbar}\,{\rm tr}_{\rm ph}\lim_{t_k\to+\infty}\\&\qquad\times\int\frac{{\rm d}\omega_k}{2\pi\hbar}e^{-i\omega_k t_k/\hbar}\,{\rm d}t'_\pi e^{i(\omega_\pi+\omega_k)t'_\pi/\hbar}\\&\qquad\times\hat{G}_{\rm ph}\left(E_k+\frac{1}{2}\omega_k\right)\hat{F}^\dagger\hat{\rho}^{(k-1)}(t'_\pi,E_\pi+E_k;E)\\&\qquad\times\hat{F}\hat{G}_{\rm ph}^\dagger\left(E_k-\frac{1}{2}\omega_k\right)\hat{G}_\pi^{(k)\dagger}\left(E_\pi-\frac{1}{2}\omega_\pi\right).\end{aligned}\tag{C.2}$$

The structure of the ω_k integral is as follows:

$$W=\int\frac{{\rm d}\omega}{2\pi\hbar}e^{-i\omega t/\hbar}\hbar^2\hat{G}_{\rm ph}\left(E+\frac{1}{2}\omega\right)\hat{A}\hat{G}_{\rm ph}^\dagger\left(E-\frac{1}{2}\omega\right),\tag{C.3}$$

where $\omega=\omega_k$, $t=t_k-t'_\pi$, $E=E_k$ and $\hat{A}$ do not commute with $\hat{G}_{\rm ph}$. By using the time representation of the propagators

$$i\hbar\hat{G}_{\rm ph}(E)=\int{\rm d}t\theta(t)\,e^{it(E-\hat{H}_{\rm ph}+i\eta)/\hbar}\tag{C.4}$$

and integrating ω eq. (C.3) becomes

$$\begin{aligned}W=&\int{\rm d}t_1\,{\rm d}t_2\theta(t_1)\theta(t_2)\delta\left(t-\frac{t_1+t_2}{2}\right)\\&\times e^{it_1(E-\hat{H}_{\rm ph})/\hbar}\hat{A}\,e^{-it_2(E-\hat{H}_{\rm ph})/\hbar}\\=&\int{\rm d}\tau\theta\left(t-\left|\frac{\tau}{2}\right|\right)e^{i(\frac{\tau}{2}+t)(E-\hat{H}_{\rm ph})/\hbar}\\&\times\hat{A}\,e^{i(\frac{\tau}{2}-t)(E-\hat{H}_{\rm ph})/\hbar}\end{aligned}\tag{C.5}$$

and using the cyclic property of the trace

$$\lim_{t\to+\infty}{\rm tr}_{\rm ph}\,W={\rm tr}_{\rm ph}\,2\pi\hbar\delta(E-\hat{H}_{\rm ph})\hat{A}\tag{C.6}$$

from which eq. (6.11) follows.

Let us now prove eq. (6.12) or equivalently,

$$\sum_k{\rm tr}\int\frac{{\rm d}E_\pi}{2\pi\hbar}\omega_\pi\hat{\rho}^{(k)}(\omega_\pi,E_\pi;E)=0.\tag{C.7}$$

Our starting point is the Schrödinger equation (6.13). There the part with $\hat{H}_\pi^{(k)}$, $\hat{H}_\pi^{(k)\dagger}$ contains the annihilation of the pion from $\hat{\rho}^{(k)}$, which go to $\hat{\rho}^{(k+1)}$, while the other part contains the transition $k-1\to k$. Then it is enough to prove that the pions that disappear in $\hat{\rho}^{(k)}$ appear in

$\hat\rho^{(k+1)}$: eq. (C.7) is a consequence of

$$\int \frac{\mathrm{d}E_\pi}{2\pi\hbar}\,\mathrm{tr}\left[\hat H_\pi^{(k)}\Big(E_\pi+\frac{1}{2}\omega_\pi\Big)\hat\rho^{(k)}(\omega_\pi,E_\pi;E)\right.$$
$$\left.-\hat\rho^{(k)}(\omega_\pi,E_\pi;E)\hat H_\pi^{(k)\dagger}\Big(E_\pi-\frac{1}{2}\omega_\pi\Big)\right]$$
$$-\int\frac{\mathrm{d}E_\pi}{2\pi\hbar}\frac{\mathrm{d}E'_\pi}{2\pi\hbar}\,\mathrm{tr}\left\{2\pi\hbar\delta\big(E'_\pi-E_\pi-\hat H_{\rm ph}\big)\right.$$
$$\times\left[\hat G_\pi^{(k+1)}\Big(E_\pi+\frac{1}{2}\omega_\pi\Big)\hat F^\dagger\hat\rho^{(k)}(\omega_\pi,E'_\pi;E)\hat F\right.$$
$$\left.\left.-\hat F^\dagger\hat\rho^{(k)}(\omega_\pi,E'_\pi;E)\hat F\hat G_\pi^{(k+1)\dagger}\Big(E_\pi-\frac{1}{2}\omega_\pi\Big)\right]\right\}=0. \quad \text{(C.8)}$$

By using the cyclic property of the trace and exchanging E_π, E'_π in the second part, eq. (C.8) follows from

$$i\hbar\Big[\hat G_\pi^{(k+1)}\circ\hat G_{\rm ph}^{(k+1)}\Big]\Big(E_\pi+\frac{1}{2}\omega_\pi\Big)$$
$$-\int\frac{\mathrm{d}E'_\pi}{2\pi\hbar}2\pi\hbar\Big(E_\pi-E'_\pi-\hat H_{\rm ph}^{(k+1)}\Big)$$
$$\times\hat G_\pi^{(k+1)}\Big(E'_\pi+\frac{1}{2}\omega_\pi\Big)=0, \quad \text{(C.9)}$$

which is equivalent to eq. (B.4).

Appendix D.

Here we wish to write the corresponding equations when an absorption mechanism is included in the model of eq. (5.2). We simply add a π-ph "vertex" in $\hat H_{\rm I}$,

$$\hat H_{\pi\,\rm ph}=\hat K+\hat K^\dagger$$
$$=\int \mathrm{d}^3x\,\mathrm{d}^3y\,K(\boldsymbol{x},\boldsymbol{y})\hat\phi_\pi^\dagger(\boldsymbol{x})\,\hat\phi_{\rm ph}(\boldsymbol{y})+\mathrm{h.c.} \quad \text{(D.1)}$$

The vertices $\hat K$, $\hat K^\dagger$ indicate that a pion can transform into a ph and vice versa. Combined with quasi-elastic, this implies that even if we start with one pion, time evolution will produce states with no pions, and states with many pions. Below the threshold this many-pion (more than one pion) states can only be virtual and the exposition is greatly simplified by not considering them. The equations generalizing (5.6) are

$$E\big|\pi(\mathrm{ph})^k\big\rangle=\hat H_0\big|\pi(\mathrm{ph})^k\big\rangle+\hat F^\dagger\big|\pi(\mathrm{ph})^{k-1}\big\rangle$$
$$+\hat F\big|\pi(\mathrm{ph})^{k+1}\big\rangle+\hat K\big|(\mathrm{ph})^{k+1}\big\rangle,$$
$$E\big|(\mathrm{ph})^k\big\rangle=\hat H_0\big|(\mathrm{ph})^k\big\rangle+\hat K^\dagger\big|\pi(\mathrm{ph})^{k-1}\big\rangle. \quad \text{(D.2)}$$

By following the same steps as before, and again keeping only direct graphs, it can be seen that all the formulae related to states $|\pi(\mathrm{ph})^k\rangle$ remain unchanged, except (6.4b):

$$\hat H_\pi^{(k)}=i\hbar\hat F\Big[\hat G_\pi^{(k+1)}\circ\hat G_{\rm ph}^{(k+1)}\Big]\hat F^\dagger+\hat K\,G_{\rm ph}^{(k+1)}\hat K^\dagger, \quad \text{(D.3)}$$

the pionic Hamiltonian now contains ph's as self-energy. In this sense, the $|\pi(\mathrm{ph})^k\rangle$ states are autonomous, however, they act as a source for $|(\mathrm{ph})^{k+1}\rangle$ states:

$$\hat\rho_A^{(k)}(\omega_k,E_k;E)=\hat G_{\rm ph}\Big(E_k+\frac{1}{2}\omega_k\Big)\hat K^\dagger\hat\rho^{(k-1)}$$
$$\times(\omega_k,E_k;E)\hat K\hat G_{\rm ph}^\dagger\Big(E_k-\frac{1}{2}\omega_k\Big), \quad \text{(D.4)}$$

where $\hat\rho_A^{(k)}$ contains the degrees of freedom of the k-th ph only (recall that the other $k-1$ ph's are traced in $\hat\rho^{(k-1)}$ which is purely pionic, and we include direct graphs only). As usual, we can write an evolution equation for $\hat\rho_A$, by computing $\hat G_{\rm ph}^{-1}\hat\rho_A-\hat\rho_A\hat G_{\rm ph}^{\dagger-1}$. Again taking trace over ph and integrating out E_k, we obtain (using the trace cyclic property)

$$i\hbar\partial_t N_A^{(k)}(t,E)=\mathrm{tr}\int\frac{\mathrm{d}E_k}{2\pi\hbar}\frac{\mathrm{d}\omega_k}{2\pi\hbar}\,\mathrm{d}t'e^{-i\omega_k(t-t')/\hbar}$$
$$\times\hat K^\dagger\hat\rho^{(k-1)}(t',E_k;E)$$
$$\times\hat K\left(\hat G_{\rm ph}^\dagger\Big(E_k-\frac{1}{2}\omega_k\Big)-\hat G_{\rm ph}\Big(E_k+\frac{1}{2}\omega_k\Big)\right), \quad \text{(D.5)}$$

where $N_A^{(k)}(t,E)=\int\mathrm{d}E_k/2\pi\hbar\,\mathrm{tr}\,\hat\rho_A^{(k)}(t,E_k;E)$ denotes the number of pions absorbed by k ph until time t. The total number of pions absorbed is obtained by integrating over t both sides of (D.5)

$$i\hbar N_A^{(k)}(\infty,E)=\int\mathrm{d}t\,\mathrm{tr}\int\frac{\mathrm{d}E_k}{2\pi\hbar}\hat K^\dagger\hat\rho^{(k-1)}(t,E_k;E)$$
$$\times\hat K2\pi i\delta\big(E_k-\hat H_{\rm ph}\big), \quad \text{(D.6)}$$

from which we read off the *effective* absorption rate

$$i\hbar\partial_t N_A^{(k)}(t,E)=\mathrm{tr}\int\frac{\mathrm{d}E_\pi}{2\pi\hbar}\hat K^\dagger\hat\rho^{(k-1)}(t,E_k;E)$$
$$\times\hat K2\pi i\delta\big(E_k-\hat H_{\rm ph}\big). \quad \text{(D.7)}$$

The difference between the exact, (D.5), and the effective, (D.7), absorption rates are due to quantum fluctuations which do not contribute to the final cross-section. The simplification occurs because the ph's are not allowed to further interact after their creation in a quasi-elastic step, and so they can directly be taken on their mass shell (as comes out of the formula). For pions such a simple result does not follow because the pions may always have further quasi-elastic (or absorption) steps.

Note the odd counting in powers of $\hbar$ in eq. (D.7). However, the limit $\hbar\to0$ will be meaningful assuming that $\hat K$ is of order $\hbar$. The reason for this is that actually the ph state is not elementary, as in our model, but rather it is formed by a nucleon and its hole, and as a consequence the field $\hat\phi_{\rm ph}(x)$ is composite containing one loop which gives a $\hbar$ factor to $\hat K$.

The equation similar to (7.9b) will be

$$\partial_t N_A=P\rho, \quad \text{(D.8)}$$

where the superoperator P is given in (D.7), and the equation similar to (7.15) for absorption is

$$\partial_t N_A = P(1 - P_V M)^{-1} af := Af. \qquad \text{(D.9)}$$

Paralleling the case of quasi-elastic, this formula indicates that after integration of virtual pions there will be effective N-body absorption mechanisms $A^{(k,k')}$ for the real pions.

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